HW5

December 10, 2025

Durrett: 4.1.3, 4.1.5, 4.2.6, 4.2.9, 4.3.3., 4.3.12, 4.4.3, 4.4.8, 4.6.7, 4.8.3

Exercise 1 Let X be indepenpendent of $\mathcal{G}$ and $Y \in \mathcal{G}$. Show that for any bounded measurable function $h : \mathbb{R}^2 \to \mathbb{R}$, the function $\varphi_h(t) := \mathsf{E}h(X,t)$ is a measurable, and

$$\mathsf{E}\big[h(X,Y) \mid \mathcal{G}\big] = \varphi_h(Y), \quad \text{a.s.}$$

Hint: start with h of the form $h(x,y) = \mathbb{1}_A(x)\mathbb{1}_B(y)$.

Exercise 2 Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration. Recall that T is a stopping time if $\{T \leq n\} \in \mathcal{F}_n$ for all n.

1. Show that a constant time $T \equiv m$ is a stopping time.
2. Show that if T and S are stopping times, then $T \wedge S := \min(T,S)$ is also a stopping time.

Exercise 3 Let X be bounded and $\mathsf{E}|Y| < \infty$. Show that

$$\mathsf{E}\Big(\mathsf{E}\big[X \mid \mathcal{G}\big]Y\Big) = \mathsf{E}\Big(\mathsf{E}\big[Y \mid \mathcal{G}\big]X\Big).$$

Note that you are required to verify all conditional expectations and expectations are well-defined.

Hint: both sides are equal to $\mathsf{E}\Big(\mathsf{E}\big[X \mid \mathcal{G}\big] \cdot \mathsf{E}\big[Y \mid \mathcal{G}\big]\Big)$.