

HW5

November 27, 2025

Durrett: 3.4.2, 3.4.4, 3.4.5, 3.4.9, 3.4.10, 3.4.12, 3.10.5, 3.10.8,

Exercise 1 1. Recall that X has the Cauchy distribution if it has density

$$f(x) = \frac{1}{\pi} \frac{1}{x^2 + c^2}, \quad c > 0.$$

Compute the ch.f. of X .

Hint: It is easier to use the residue theorem from complex analysis.

2. Show that if $X_1, \dots, X_n$ are i.i.d. Cauchy, then $\frac{S_n}{n}$ and X_1 have the same distribution.

Hint: compute the ch.f.

3. Let X, Y be independent $\mathcal{N}(0, 1)$. Show that $Z = \frac{X}{Y}$ has the Cauchy distribution.

Hint: it suffices to compute $P(X \geq aY)$, where you can use that $X - aY$ is Gaussian.

Exercise 2 Let X_n be i.i.d., with a continuous density $f(x)$, supported on $[-1, 1]$, with $f(0) > 0$.

1. Show that

$$\lim_{n \rightarrow \infty} n \int_{\mathbb{R}} \left[\cos \frac{\xi}{nx} - 1 \right] f(x) dx \rightarrow -c|\xi|$$

for some constant $c > 0$.

2. Show that for every ξ ,

$$\mathbb{E} e^{i\xi \sum_{m=1}^n \frac{1}{nX_m}} = \exp \left(n \log \int_{\mathbb{R}} \cos \frac{\xi}{nx} f(x) dx \right) \rightarrow e^{-c|\xi|}.$$

Hence, $\frac{1}{n} \left(\frac{1}{X_1} + \dots + \frac{1}{X_n} \right) \Rightarrow \text{Cauchy distribution}$.

Exercise 3 Let $X_1, X_2, \dots$ be independent random vectors in $\mathbb{R}^d$, with symmetric distribution, that is, X_n and $-X_n$ have the same distribution. Let $S_n = X_1 + \dots + X_n$ be the partial sum.

The goal is to establish the following generalization of Kolmogorov's maximal inequality:

$$P \left(\max_{1 \leq k \leq n} |S_k| \geq r \right) \leq 2P(|S_n| \geq r), \quad \forall r > 0. \quad (0.1)$$

where $|\cdot|$ is the Euclidean norm.

1. Use symmetry and independence to show that (S_k, S_n) and $(S_k, 2S_k - S_n)$ have the same distribution for every $k \geq 0$.

2. Let $T = \min\{k : |S_k| \geq r\}$. Show that

$$\mathbf{P}(T = k, |S_n| < r) \leq \mathbf{P}(|S_k| \geq r, |S_n| < r) \leq \mathbf{P}(|S_k| \geq r, |S_n| \geq r).$$

3. Conclude the proof of (0.1) using

$$\mathbf{P}(T \leq n) \leq \mathbf{P}(|S_n| \geq r) + \sum_{k=1}^n \mathbf{P}(T = k, |S_n| < r).$$

Remark: in fact, our argument does not rely on any specific properties of $\mathbb{R}^d$. For example, consider $X_n = \xi_n \cdot e^{2n\pi i x} \in \mathcal{C}[0, 1]$, where ξ_n are i.i.d. $\mathcal{N}(0, 1)$. With additional effort, one can show that $\sum_{n=1}^{\infty} \xi_n e^{2n\pi i x}$ converges almost surely in $\mathcal{C}[0, 1]$. This leads to another representation of the Brownian motion.