HW3

October 30, 2025

Durrett: 2.2.2, 2.2.4, 2.2.7, 2.3.8, 2.3.9, 2.4.2

In the next two problems we use λ to denote the Lebesgue measure on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$.

Exercise 1 Let X_n, $n \geq 1$, be r.v.s on $(\Omega, \mathcal{F}, \mathsf{P})$ with $\mathsf{E}|X_n| < \infty$. Let $\mathcal{A}$ be all subsets of $\mathbb{N} = \{1, 2, \dots\}$, and μ be the counting measure on $(\mathbb{N}, \mathcal{A})$, that is,

$$\mu(A) = \text{ number of elements in } A, \quad A \in \mathcal{A}.$$

On the product space $(\Omega \times \mathbb{N}, \mathcal{F} \otimes \mathcal{A}, \mathsf{P} \times \mu)$, show that,

1. the map $\mathbf{X}(\omega, n) = X_n(\omega)$ is $(\mathcal{F} \otimes \mathcal{A})$-measurable;
2. if $\sum_{n=1}^{\infty} \mathsf{E}|X_n| < \infty$, then

$$\int_{\Omega \times \mathbb{N}} \mathbf{X}(\omega, n)(\mathsf{P} \times \mu)(d\omega dn) = \int_{\Omega} \sum_{n=1}^{\infty} X_n(\omega)\, \mathsf{P}(d\omega) = \sum_{n=1}^{\infty} \int_{\Omega} X_n(\omega)\, \mathsf{P}(d\omega).$$

Exercise 2 Let $X \geq 0$ be a r.v. on $(\Omega, \mathcal{F}, \mathsf{P})$. On the product space $(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathsf{P} \times \lambda)$, show that,

1. $\{(\omega, y) : 0 \leq y \leq X(\omega)\} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$;
   Hint: the map $(x, y) \mapsto x - y$ *is measurable;*
2. the following equality holds:

$$\int_{\Omega} X(\omega)\, \mathsf{P}(d\omega) = (\mathsf{P} \times \lambda)\big(\{(\omega, y) : 0 \leq y \leq X(\omega)\}\big) = \int_0^{\infty} \mathsf{P}(X \geq y)\, \lambda(dy).$$

Conclude that

$$\mathsf{E}X \leq \sum_{n=0}^{\infty} \mathsf{P}(X \geq n).$$

Exercise 3 Use Fubini's Theorem to show that if $X \geq 0$ and $p > 0$, then

$$\mathsf{E}X^p = \int_0^{\infty} p y^{p-1}\, \mathsf{P}(X > y)\, dy.$$

Hint: $x^p = \int_0^x p y^{p-1}\, dy$, $x \geq 0$.

Exercise 4 Let $(X_n)_{n\geq 1}$ be independent. Show that

$$\sup_n X_n < \infty, \text{ a.s.} \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \mathsf{P}(X_n > A) < \infty \text{ for some } A > 0.$$

Hint: for the "$\Leftarrow$" direction use the first Borel–Cantelli; for the "$\Rightarrow$" direction, use proof by contradiction and show $\sup_n X_n \geq A$ *a.s. for every* A *using the second Borel–Cantelli when the condition does not hold.*

Exercise 5 Recall from the "St. Petersburg game" we have the i.i.d. r.v.'s $(X_n)_{n\geq 1}$ with distribution

$$\mathsf{P}(X_1 = 2^j) = 2^{-j}, \quad j \geq 1.$$

1. Show that for every $M > 0$, $\sum_{n=1}^{\infty} \mathsf{P}(X_n \geq Mn \log_2 n) = \infty$.
2. Show that for every $M > 0$, $\mathsf{P}(\{\omega : X_n(\omega) \geq Mn \log_2 n, \text{ i.o.}\}) = 1$.
3. Show that for every $M > 0$, $\limsup_{n\to\infty} \frac{X_n}{n \log_2 n} \geq M$ a.s.
4. Show that $\limsup_{n\to\infty} \frac{S_n}{n \log_2 n} = \limsup_{n\to\infty} \frac{X_n}{n \log_2 n} = \infty$ a.s.