HW2

October 15, 2025

Durrett: 1.3.3, 1.4.1, 1.6.4, 1.6.8, 1.6.15, 2.1.1, 2.3.4, 2.3.6, 2.3.7,

Exercise 1 Show that if $X_n \to X$ and $Y_n \to Y$ both in probability, then $X_n \pm Y_n \to X \pm Y$, $X_n \cdot Y_n \to X \cdot Y$ all in probability.

Exercise 2 Given two r.v.s X and Y, define

$$\rho_1(X,Y) = \inf\{\varepsilon > 0 : \mathsf{P}(|X-Y| > \varepsilon) \le \varepsilon\}, \quad \rho_2(X,Y) = \inf\{\mathsf{P}(|X-Y| > \varepsilon) + \varepsilon : \varepsilon > 0\}.$$

1. Show that each of ρ_1, ρ_2 defines a metric on the space of r.v.s, i.e.,
   - $\rho_i(X,Y) \ge 0$, where equality holds if and only if $X = Y$ a.s.
   - $\rho_i(X,Y) + \rho_i(Y,Z) \ge \rho_i(X,Z)$.
2. Show that, for $i = 1, 2$, $X_n \to X$ in probability if and only if $\rho_i(X_n, X) \to 0$.

Exercise 3 Let X and Y be two independent r.v.s on $(\Omega, \mathcal{F}, \mathsf{P})$.

1. Show that $h_1(X)$ and $h_2(Y)$ are independent for all Borel measurable functions $h_i : \mathbb{R} \to \mathbb{R}$.

   Hint: use an appropriatedly chosen definition for independence!

2. Let f, g be Borel measurable functions such that $\mathsf{E}|f(X)| < \infty$, $\mathsf{E}|g(Y)| < \infty$. Show that

$$\mathsf{E}f(X)g(Y) = \mathsf{E}f(X) \cdot \mathsf{E}g(Y).$$

   Hint: start from bounded Borel functions f and g, and then approximate.