HW1

September 18, 2025

Durrett: 1.1.1, 1.1.2, 1.1.3, 1.1.4, 1.1.5, 1.2.1, 1.2.4

Exercise 1 Let $(\Omega, \mathcal{F}_0, \mathsf{P}_0)$ be a probability space. We say that $A \subset \Omega$ is a P_0-*null set* (which may or may not be an element of $\mathcal{F}_0$), if there exists $N \in \mathcal{F}_0$ such that $A \subset N$ and $\mathsf{P}_0(N) = 0$. Denote by $\mathcal{N}$ the collection of all P_0-null sets.

1. Let
$$\mathcal{F} = \{A \subset \Omega : \exists B_1, B_2 \in \mathcal{F}_0 \text{ s.t. } B_1 \subset A \subset B_2,\ A \setminus B_1, B_2 \setminus A \in \mathcal{N}\}.$$
Show that $\mathcal{F}$ is a σ-algebra, and it is the smallest σ-algebra containing $\mathcal{F}_0$ and $\mathcal{N}$.

2. Let $\mathsf{P} : \mathcal{F} \to [0, 1]$ be defined by $\mathsf{P}(A) = \mathsf{P}_0(B_1)$ where $A \setminus B_1 \in \mathcal{N}$ and $B_1 \in \mathcal{F}_0$. Show that this definition is independent of the choice of B_1.

3. Show that $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space. (This is called the *completion* of $(\Omega, \mathcal{F}_0, \mathsf{P}_0)$.)

Exercise 2 We first give two definitions.
We say that $\mathcal{A}$ is a π-*system* if it is closed under intersection, that is $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$.
We say that $\mathcal{D}$ is a λ-*system* if

- $\Omega \in \mathcal{D}$,
- $A, B \in \mathcal{D},\ A \subset B \implies B \setminus A \in \mathcal{D}$,
- $A_n \uparrow A,\ A_n \in \mathcal{D} \implies A \in \mathcal{D}$.

Since any intersection of λ-systems is still a λ-system, we can define the *smallest* λ-system generated by an arbitrary collection of set $\mathcal{A}$, denoted by $\lambda(\mathcal{A})$.

1. Show that $\mathcal{A}$ is a σ-algebra if and only if it is both a π-system and a λ-system.

2. Use the *method of appropriate sets* to show that if $\mathcal{A}$ is a π-system, then $\sigma(\mathcal{A}) = \lambda(\mathcal{A})$.

Exercise 3 Let μ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We say that $x \in \operatorname{supp} \mu$ (the support of μ), if $\mu(x - \varepsilon, x + \varepsilon) > 0$ for every $\varepsilon > 0$.

1. Show that if $\mu\{x\} > 0$, then $x \in \operatorname{supp} \mu$.

2. Show that if $\mu = \mu_X$ is the distribution of a continuous r.v. X with continuous density f, and $f(x) > 0$, then $x \in \operatorname{supp} \mu$.

3. Show that $(\operatorname{supp} \mu)^c$ is an open set, that is, if $x \notin \operatorname{supp} \mu$, then there is $\delta > 0$ such that $(x - \delta, x + \delta) \notin \operatorname{supp} \mu$.

   As a consequnce, $\operatorname{supp} \mu$ is always a closed set.

4. Recall that the Cantor set is defined by $\mathcal{C} = [0,1] \setminus \bigcup_{n\geq 1,\ 1\leq k\leq 2^{n-1}} I_n^{(k)}$, where

$$I_1^{(1)} = (\frac{1}{3}, \frac{2}{3}), \quad I_2^{(1)} = (\frac{1}{9}, \frac{2}{9}), \quad I_2^{(2)} = (\frac{7}{9}, \frac{8}{9}), \quad \ldots$$

and the definition of Cantor function φ (see, e.g., Example 1.2.7 in Durrett). The distribution function φ defines a probability measure $\mu = \mu_\varphi$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(a) Show that $\mathcal{C}$ is a closed set and

$$\mathcal{C} = \text{closure of} \bigcup_{n\geq 1,\ 1\leq k\leq 2^{n-1}} \partial I_n^{(k)}.$$

(b) Show that

$$\mathcal{C} = \Big\{ \sum_{n=1}^{\infty} \frac{2\cdot \varepsilon_n}{3^n},\ \varepsilon_n \in \{0,1\} \Big\}.$$

(c) Show that $x \in \operatorname{supp} \mu_\varphi$ for every $x \in \partial I_n^{(k)}$.

(d) Show that $\operatorname{supp} \mu_\varphi = \mathcal{C}$.