HW7

November 13, 2024

Exercise 1 Let $X \sim \text{Poi}(\lambda)$ and $Y \sim \text{Poi}(\mu)$ be independent. Show that $Z = X + Y \sim \text{Poi}(\lambda + \mu)$.

Hint: you can either compute the ch.f. or the probability mass function.

Exercise 2 Let ξ_n, $n \geq 1$, be i.i.d. $\text{Ber}(p)$ r.v.'s, and let $N \sim \text{Poi}(\lambda)$ be independent with ξ_n. Let

$$X = \sum_{n=1}^{N} \xi_n, \quad Y = \sum_{n=1}^{N} (1 - \xi_n).$$

Compute the joint probability mass function $\mathsf{P}(X = k,\ Y = m)$ and show that X and Y are independent Poisson with parameters $p\lambda$ and $(1-p)\lambda$.

Exercise 3 Let $X \sim \mathcal{N}(0, 1)$. Show that $Y = 1/X^2$ has the density

$$f(y) = \frac{1}{\sqrt{2\pi y^3}} e^{-\frac{1}{2y}}.$$

(In fact, Y is a stable law with index $1/2$.)

Exercise 4

1. Recall that X has the Cauchy distribution if it has density

   $$f(x) = \frac{1}{\pi} \frac{1}{x^2 + c^2}, \quad c > 0.$$

   Compute the ch.f. of X.

2. Show that if $X_1, \ldots, X_n$ are i.i.d. Cauchy, then $\dfrac{S_n}{n}$ and X_1 have the same distribution.

   Hint: compute the ch.f.'s.

3. Let X, Y be independent $\mathcal{N}(0, 1)$. Show that $Z = \dfrac{X}{Y}$ has the Cauchy distribution.

   Hint: it suffices to compute $\mathsf{P}(X \geq aY)$. Note that $X - aY$ is also normal.

Exercise 5 Let X_n be i.i.d., with a continuous density $f(x)$, supported on $[-1, 1]$, with $f(0) > 0$.

1. Show that

   $$\lim_{n\to\infty} n \int_{\mathbb{R}} \left[\cos \frac{\xi}{nx} - 1\right] f(x)\, dx \to -c|\xi|$$

   for some constant $c > 0$.

2. Show that for every ξ,

   $$\mathsf{E} e^{i\xi \sum_{m=1}^{\infty} \frac{1}{nX_m}} = \exp\Big(n \log \int_{\mathbb{R}} \cos \frac{\xi}{nx} f(x)\, dx\Big) \to e^{-c|\xi|}.$$

   Hence, $\dfrac{1}{n}\left(\dfrac{1}{X_1} + \cdots + \dfrac{1}{X_n}\right) \Rightarrow$ Cauchy distribution.

Exercise 6 Let $X_1, X_2, \dots$ be independent random vectors in $\mathbb{R}^d$, with symmetric distribution, that is, X_n and $-X_n$ have the same distribution. Let $S_n = X_1 + \cdots + X_n$ be the partial sum.

The goal is to establish the following generalization of Kolmogorov's maximal inequality:

$$\mathsf{P}\Big(\max_{1\le k\le n} |S_k| \ge r\Big) \le 2\mathsf{P}(|S_n| \ge r), \quad \forall r > 0. \tag{1}$$

where $|\cdot|$ is the Euclidean norm.

1. Use symmetry and independence to show that (S_k, S_n) and $(S_k, 2S_k - S_n)$ have the same distribution for every $k \ge 0$.

2. Let $T = \min\{k : |S_k| \ge r\}$. Show that

$$\mathsf{P}(T = k,\ |S_n| < r) \le \mathsf{P}(|S_k| \ge r,\ |S_n| < r) \le \mathsf{P}(|S_k| \ge r,\ |S_n| \ge r).$$

3. Conclude the proof of Eq. (1) using

$$\mathsf{P}(T \le n) \le \mathsf{P}(|S_n| \ge r) + \sum_{k=1}^{n} \mathsf{P}(T = k,\ |S_n| < r).$$

Remark: in fact we did not use anything about $\mathbb{R}^d$*. For example, let* $X_n = \xi_n \cdot e^{2n\pi i x} \in \mathcal{C}[0,1]$*, where* ξ_n *are i.i.d.* $\mathcal{N}(0,1)$*. With a little more effort one can show that* $\sum_{n=1}^{\infty} \xi_n e^{2n\pi i x}$ *converges almost surely in* $\mathcal{C}[0,1]$*. This leads to another representation of the Brownian motion.*