HW4

October 15, 2024

Exercise 1 Use Fubini's Theorem to show that if $X \geq 0$ and $p > 0$, then

$$\mathsf{E}X^p = \int_0^\infty p y^{p-1}\, \mathsf{P}(X > y)\, dy.$$

Hint: $x^p = \int_0^x p y^{p-1}\, dy$, $x \geq 0$.

Exercise 2 Prove the following general version of L^2 weak law of large numbers. Let $(X_n)_{n\geq 1}$ be independent r.v.'s with

$$\mathsf{E}X_n = 0, \quad \mathsf{E}X_n X_m \leq \phi(n-m),\ n \geq m,$$

where $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ satisfies $\lim_{k\to\infty} \phi(k) = 0$. Show that

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \to 0, \qquad \text{in probability.}$$

Hint: estimate $\mathrm{Var}(S_n)$ *using* $\phi(k)$ *and show that* $\mathrm{Var}(S_n)/n^2 \to 0$.

Exercise 3 Let $(X_n)_{n\geq 1}$ be i.i.d. with

$$\mathsf{P}\big(X_1 = (-1)^k k\big) = \frac{c}{k^2 \log k}, \quad k \geq 2,$$

where $c = \Big[\sum_{k=2}^\infty \frac{1}{k^2 \log k}\Big]^{-1} > 0$ is the normalizing constant (its exact value is irrelevant for us).

1. Show that $\mathsf{E}|X_1| = \infty$.
2. Find a constant μ so that $S_n/n \to \mu$ in probability.

Exercise 4 Let $(X_n)_{n\geq 1}$ be independent. Show that

$$\sup_n X_n < \infty, \text{ a.s.} \quad \Leftrightarrow \quad \sum_{n=1}^\infty \mathsf{P}(X_n > A) < \infty \text{ for some } A > 0.$$

Hint: for the "⇐" direction use the first Borel–Cantelli; for the "⇒" direction, use proof by contradiction and show $\sup_n X_n \geq A$ *a.s. for every* A *using the second Borel–Cantelli when the condition does not hold.*

Exercise 5 Recall from the "St. Petersburg game" we have the i.i.d. r.v.'s $(X_n)_{n\geq 1}$ with distribution

$$\mathsf{P}(X_1 = 2^j) = 2^{-j}, \quad j \geq 1.$$

1. Show that for every $M > 0$, $\sum_{n=1}^{\infty} \mathsf{P}(X_n \geq Mn \log_2 n) = \infty$.
2. Show that for every $M > 0$, $\mathsf{P}(\{\omega : X_n(\omega) \geq Mn \log_2 n, \text{ i.o.}\}) = 1$.
3. Show that for every $M > 0$, $\limsup_{n\to\infty} \frac{X_n}{n \log_2 n} \geq M$ a.s.
4. Show that $\limsup_{n\to\infty} \frac{S_n}{n \log_2 n} = \limsup_{n\to\infty} \frac{X_n}{n \log_2 n} = \infty$ a.s.

Exercise 6 Recall from the coupon collector problem, we have $\xi_1, \xi_2, \ldots$ be i.i.d. uniform on $\{1, 2, \ldots, n\}$, and

$$\tau_k^n = \min\{m \geq 0 : \big|\{\xi_1, \xi_2, \ldots, \xi_m\}\big| \geq k\}, \quad 0 \leq k \leq n,$$

be the *first* time that one collects k *distinct* coupons (noting that $\tau_0^n = 0$).

For any fixed $r_1, \cdots, r_n \geq 1$, by counting, compute directly the probability

$$\mathsf{P}\big(X_{n,k} = r_k,\ 1 \leq k \leq n\big).$$

Conclude that $X_{n,k}$ are independent geometric r.v.'s with parameter $p_k = \frac{n-k+1}{n}$, $1 \leq k \leq n$. Note that $Y \sim \text{Geo}(p)$ if $\mathsf{P}(Y = r) = (1-p)^{r-1}p$, $r \geq 1$.

Hint: this can be reduced to computing the size of a certain subset of $\{1, 2, \ldots, n\}^{r_1+\cdots+r_n}$.

Exercise 7 Let X_n, $n \geq 1$, be i.i.d. r.v.'s with c.d.f. F. Recall that the empirical distribution function from the first n samples is given by

$$F_n^\omega(x) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{\{X_k(\omega)\leq x\}}, \quad x \in \mathbb{R}.$$

1. Explain carefully why $F_n^\omega(x)$ is a r.v. for each fixed $x \in \mathbb{R}$ and $n \geq 1$.

   Hint: $\mathbb{1}_{\{X_k(\omega)\leq x\}} = \mathbb{1}_{(-\infty,x]}\big(X_k(\omega)\big)$ *and* $\mathbb{1}_{(-\infty,x]}(\cdot)$ *is a Borel measurable function on* $\mathbb{R}$.
2. Show that for every ω and $n \geq 1$, the function $x \mapsto F_n^\omega(x)$ is increasing and right continuous.
3. Show that for every ω,

   $$\sup_{x\in\mathbb{R}} |F_n^\omega(x) - F(x)| = \sup_{q\in\mathbb{Q}} |F_n^\omega(q) - F(q)|.$$

   Hint: use right continuity of F_n *and* F.
4. Show that

   $$Y_n(\omega) = \sup_{x\in\mathbb{R}} |F_n^\omega(x) - F(x)|$$

   is a random variable.