HW3

October 8, 2024

In this problem set we use λ to denote the Lebesgue measure on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$.

Exercise 1 Let X_n, $n \geq 1$, be r.v.'s on $(\Omega, \mathcal{F}, \mathsf{P})$ with $\mathsf{E}|X_n| < \infty$. Let $\mathcal{A}$ be all subsets of $\mathbb{N} = \{1, 2, \dots\}$, and μ be the counting measure on $(\mathbb{N}, \mathcal{A})$, i.e.,

$$\mu(A) = \text{ number of elements in } A, \quad A \in \mathcal{A}.$$

On the product space $(\Omega \times \mathbb{N}, \mathcal{F} \otimes \mathcal{A}, \mathsf{P} \times \mu)$, show that,

1. the map $\mathbf{X}(\omega, n) = X_n(\omega)$ is $(\mathcal{F} \otimes \mathcal{A})$-measurable;
2. if $\sum_{n=1}^{\infty} \mathsf{E}|X_n| < \infty$, then

$$\int_{\Omega \times \mathbb{N}} \mathbf{X}(\omega, n)(\mathsf{P} \times \mu)(d\omega dn) = \int_{\Omega} \sum_{n=1}^{\infty} X_n(\omega)\, \mathsf{P}(d\omega) = \sum_{n=1}^{\infty} \int_{\Omega} X_n(\omega)\, \mathsf{P}(d\omega).$$

Exercise 2 Let $X \geq 0$ be a r.v. on $(\Omega, \mathcal{F}, \mathsf{P})$. On the product space $(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathsf{P} \times \lambda)$, show that,

1. $\{(\omega, y) : 0 \leq y \leq X(\omega)\} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$; *hint: the map* $(x, y) \mapsto x - y$ *is measurable;*
2. the following equality holds:

$$\int_{\Omega} X(\omega)\, \mathsf{P}(d\omega) = (\mathsf{P} \times \lambda)\big(\{(\omega, y) : 0 \leq y \leq X(\omega)\}\big) = \int_0^{\infty} \mathsf{P}(X \geq y)\, \lambda(dy).$$

Conclude that

$$\mathsf{E}X \leq \sum_{n=0}^{\infty} \mathsf{P}(X \geq n).$$

Exercise 3 Let $f(x_1, x_2)$ be the density of the random vector (X_1, X_2), i.e.,

$$\mathsf{P}\big((X_1, X_2) \in A\big) = \int_A f(x_1, x_2)\, dx_1 dx_2, \quad \forall A \in \mathcal{B}(\mathbb{R}^2).$$

Suppose that $f(x_1, x_2) = g_1(x_1) g_2(x_2)$ where $g_1, g_2 \geq 0$ and are measurable.

1. Using Fubini's theorem to show that $c_i = \int_{\mathbb{R}} g_i(t)\, dt \in (0, \infty)$, $i = 1, 2$.
2. Show that X_1, X_2 are independent continuous r.v.'s with density $(c_i)^{-1} g_i(t)$, $i = 1, 2$.

Exercise 4 Let $(\Omega, \mathcal{F}, \mathsf{P}) = \big((0,1), \mathcal{B}\big((0,1)\big), \lambda\big)$. Let ξ_n be defined via the dyadic expansion of ω (with infinitely many 1's):

$$\omega = \sum_{n=1}^{\infty} \xi_n(\omega) \frac{1}{2^n}, \quad \xi_n(\omega) \in \{0, 1\}.$$

The goal is to show that $(\xi_n)_{n\geq 1}$ are i.i.d. Ber(1/2).

1. Let $N \geq 1$. Show that for every sequence $(\varepsilon_n)_{n=1}^N \in \{0,1\}^N$,

$$\mathsf{P}\big(\{\omega : \xi_n(\omega) = \varepsilon_n, \ 1 \leq n \leq N\}\big) = \frac{1}{2^N} = \prod_{n=1}^{N} \mathsf{P}\big(\xi_n = \varepsilon_n\big).$$

   Hint: $\overline{\xi_1 \xi_2 \cdots \xi_N}$ *is the binary representation of the integer* $[2^N \omega]$.

2. Let $N \geq 1$. Show that if $A_n \in \sigma(\xi_n)$, $1 \leq n \leq N$,

$$\mathsf{P}\Big(\bigcap_{n=1}^{N} A_n\Big) = \prod_{n=1}^{N} \mathsf{P}(A_n). \tag{1}$$

   Hint: $\sigma(\xi_n) = \big\{\varnothing, \{\xi_n = 0\}, \{\xi_n = 1\}, \Omega\big\}$.

3. Show that if $A_n \in \sigma(\xi_n)$, $n \geq 1$, then

$$\mathsf{P}\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \prod_{n=1}^{\infty} \mathsf{P}(A_n).$$

   Hint: justify taking the limit $N \to \infty$ *in Eq.* (1).

Exercise 5 Let $F : [A, B] \to [C, D]$ be an increasing function which is continuous at A and B. Define the inverse function $F^{-1} : [C, D] \to [A, B]$ by

$$F^{-1}(y) := \sup\{x : F(x) < y\}.$$

1. Show that if F is right continuous, then F^{-1} is left continuous.

2. Show that if F is left continuous, then F^{-1} is right continuous.

3. Show that if F is right continuous, then

$$\{y : F^{-1}(y) \leq x_0\} = \{y : y \leq F(x_0)\}$$

   for every $x_0 \in [A, B]$.