HW2

September 24, 2024

Exercise 1 Let A_n be measurable sets. Show that

$$\mathsf{P}\big(\limsup_{n\to\infty} A_n\big) \geq \limsup_{n\to\infty} \mathsf{P}\big(A_n\big), \quad \mathsf{P}\big(\liminf_{n\to\infty} A_n\big) \leq \liminf_{n\to\infty} \mathsf{P}\big(A_n\big).$$

Hint: use Fatou's lemma and use some results from HW1.

Exercise 2 Show that if $X_n \to X$ and $Y_n \to Y$ both in probability, then $X_n \pm Y_n \to X \pm Y$, $X_n \cdot Y_n \to X \cdot Y$ all in probability.

Exercise 3

1. Show that $X_n \to 0$ in probability if and only if $\mathsf{E}\dfrac{|X_n|}{1+|X_n|} \to 0$.
2. Let f be a bounded, uniformly continuous function on $\mathbb{R}$. Show that $X_n \to 0$ in probability implies $\mathsf{E}f(X_n) \to f(0)$.

Exercise 4

1. Let f be a continuous function on $\mathbb{R}$. Show that if $X_n \to X$ in probability, then $f(X_n) \to f(X)$ in probability.

   Hint: first consider $f_M(x) = (-M) \vee x \wedge M$.
2. Give a counterexample where f is merely Borel measurable.

Exercise 5 Given two r.v.'s X and Y, define

$$\rho_1(X,Y) = \inf\{\varepsilon > 0 : \mathsf{P}(|X-Y| > \varepsilon) \leq \varepsilon\}, \quad \rho_2(X,Y) = \inf\{\mathsf{P}(|X-Y| > \varepsilon) + \varepsilon : \varepsilon > 0\}.$$

1. Show that each of ρ_1, ρ_2 defines a metric on the space of r.v.'s, i.e.,
   - $\rho_i(X,Y) \geq 0$, where equality holds if and only if $X = Y$ a.s.
   - $\rho_i(X,Y) + \rho_i(Y,Z) \geq \rho_i(X,Z)$.
2. Show that, for $i = 1,2$, $X_n \to X$ in probability if and only if $\rho_i(X_n, X) \to 0$.

Exercise 6 Let X_n be a sequence of r.v.'s. Show that there exists a sequence of constants a_n such that $X_n/a_n \to 0$ almost surely.

Hint: use Borel–Cantelli Lemma.

Exercise 7

1. Let X_n be a Cauchy sequence in probability, i.e.,

$$\lim_{N\to\infty} \sup_{n,m\geq N} \mathsf{P}(|X_n - X_m| \geq \varepsilon) = 0, \quad \forall \varepsilon > 0.$$

   Show that there exists a r.v. X s.t. $X_n \to X$ in probability.

2. Let X_n be a Cauchy sequence in L^p, i.e.,

$$\lim_{N\to\infty} \sup_{n,m\geq N} \mathsf{E}|X_n - X_m|^p = 0.$$

   Show that there exists a r.v. X s.t. $X_n \to X$ in L^p.

Exercise 8 Let X be a r.v. Show that a r.v. Y is a measurable map from $\big(\Omega, \sigma(X)\big)$ to $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ if and only if $Y = f(X)$ where $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function.

Hint: for the "only if" direction, consider the collection $\mathcal{H}$ of all such r.v.'s Y; then $\mathcal{H}$ contains all simple r.v.'s and behaves well under limits.

Exercise 9 Let X and Y be two independent r.v.'s on $(\Omega, \mathcal{F}, \mathsf{P})$.

1. Show that $h_1(X)$ and $h_2(Y)$ are independent for all Borel measurable functions $h_i : \mathbb{R} \to \mathbb{R}$.

   Hint: choose the definition wisely and the proof can be as short as one line.

2. Let f, g be Borel measurable functions such that $\mathsf{E}|f(X)| < \infty$, $\mathsf{E}|g(Y)| < \infty$. Show that

$$\mathsf{E}f(X)g(Y) = \mathsf{E}f(X) \cdot \mathsf{E}g(Y).$$