HW10

December 20, 2024

In the first two problems, we consider a nearest neighbor random walk on $\mathbb{Z}$:

$$\mathsf{P}[X_{k+1} = X_k + 1 \,|\, X_k = n] = p_n, \quad \mathsf{P}[X_{k+1} = X_k - 1 \,|\, X_k = n] = q_n,$$

where $p_n, q_n \geq 0$ and $p_n + q_n = 1$. Recall that $(\mu_n)_{n\in\mathbb{Z}}$ is an invariant measure if $\mu\mathsf{P} = \mathsf{P}$, that is,

$$\mu_n = p_{n-1}\mu_{n-1} + q_{n+1}\mu_{n+1}, \quad \forall n \in \mathbb{Z}.$$

Exercise 1 Let μ be an invariant measure. We define the "flux" from n to $n+1$ to be

$$j_n = p_n\mu_n - q_{n+1}\mu_{n+1}.$$

1. Show that j_n is constant.
2. Show that if μ is an invariant distribution, then $j_n \equiv 0$.
3. Suppose that $p_0 = 1$ and $p_{-1} = 0$, and thus the random walk can never cross $[-1, 0]$. If $q_n, p_n > 0$ for all $n \geq 1$, then $j_n = 0$ for $n \geq 0$.

Remark: the condition $j_n = 0$ implies that

$$p_n\mu_n = q_{n+1}\mu_{n+1}.$$

This is the "detailed balance" condition.

Exercise 2 Let $q_0 = 0$ and $p_n = \dfrac{1}{2} - \dfrac{1}{20n^\alpha}$, $n \geq 1$ for some $\alpha > 0$.

1. Use the detailed balance condition to determine all invariant measures μ.
2. Find the sufficient and necessary condition in terms of α for an invariant *distribution* to exist.

In the next two problems, we assume that X_n is a simple random walk on $\mathbb{Z}^d$, that is,

$$\mathsf{P}[X_{n+1} = X_n \pm e_i \,|\, X_n] = \frac{1}{2d}, \quad i \in \{1, 2, \ldots, n\},\ e_i \text{ unit vectors in } \mathbb{Z}^d.$$

Exercise 3 Let μ be an invariant measure.

1. Show that
$$\mu_m = \frac{1}{2d}\Big[\mu_{m+e_1} + \mu_{m-e_1} + \cdots + \mu_{m+e_d} + \mu_{m-e_d}\Big]. \tag{1}$$

2. Use the first part to deduce that if there exists $m_* \in \mathbb{Z}^d$ such that

$$\mu_{m_*} = K = \sup_{m \in \mathbb{Z}^d} \mu_m,$$

then $\mu_m = K$ for all $m \in \mathbb{Z}^d$.

3. Show that μ cannot be an invariant distribution.

Exercise 4 We change notation and write $f(m) = \mu_m$ for μ satisfying (1). Assume in addition that f is bounded.

1. Show that $f(X_n)$ is a martingale.
2. Show that $f(X_n)$ converges P^x-a.s. and in $L^1(\mathsf{P}^x)$, if $X_0 = x$.
3. We know that $\lim_{n\to\infty} f(X_n)$ is measurable to the exchangable σ-algebra, and thus is a constant. Show that this constant is $f(x)$.
4. Let $\xi_n = X_n - X_{n-1}$. Almost surely we have

$$\lim_{n\to\infty} f(x + \xi_1 + \xi_2 + \cdots + \xi_n) = f(x).$$

Show that for all $k \geq 1$,

$$f(x + \xi_1 + \cdots + \xi_k) = f(x).$$

5. Conclude that f is a constant.

Remark: This means that the invariant measure of the simple random walk on $\mathbb{Z}^d$ is unique up to a factor. We can also interprete this result as a discrte version of the Liouville Theorem, that is, any bounded discrte harmonic functions must be constant.