HW1

September 15, 2024

Exercise 1 For every $A \subset \Omega$, its *indicator function* $\mathbb{1}_A : \Omega \to \{0,1\}$ is defined by

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \in A^c. \end{cases}$$

1. Suppose that $A_n \uparrow A$ or $A_n \downarrow A$. Show that

$$\lim_{n\to\infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_A(\omega), \quad \forall \omega \in \Omega.$$

2. Let $A_n \subset \Omega$ and

$$I = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m, \quad S = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

   Show that

$$\liminf_{n\to\infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_I(\omega), \quad \limsup_{n\to\infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_S(\omega), \qquad \forall \omega \in \Omega.$$

Hint: recall that for a sequence (a_n), its lower and upper limits are defined by

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 1} \inf_{m\geq n} a_m, \quad \limsup_{n\to\infty} a_n = \inf_{n\geq 1} \sup_{m\geq n} a_m;$$

or alternatively, they are the smallest and largest limit points of the set $\{a_n\}$.

Exercise 2 Let $(\Omega, \mathcal{F}_0, \mathsf{P}_0)$ be a probability space. We say that $A \subset \Omega$ is a P_0-*null set* (which may or may not be an element of $\mathcal{F}_0$), if there exists $N \in \mathcal{F}_0$ such that $A \subset N$ and $\mathsf{P}_0(N) = 0$. Denote by $\mathcal{N}$ the collection of all P_0-null sets.

1. Let

$$\mathcal{F} = \{A \subset \Omega : \exists B_1, B_2 \in \mathcal{F}_0 \text{ s.t. } B_1 \subset A \subset B_2,\ A \setminus B_1, B_2 \setminus A \in \mathcal{N}\}.$$

   Show that $\mathcal{F}$ is a σ-algebra, and it is the smallest σ-algebra containing $\mathcal{F}_0$ and $\mathcal{N}$.

2. Let $\mathsf{P} : \mathcal{F} \to [0,1]$ be defined by $\mathsf{P}(A) = \mathsf{P}_0(B_1)$ where $A \setminus B_1 \in \mathcal{N}$ and $B_1 \in \mathcal{F}_0$. Show that this definition is independent of the choice of B_1.

3. Show that $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space. (This is called the *completion* of $(\Omega, \mathcal{F}_0, \mathsf{P}_0)$.)

Exercise 3 Recall that $\mathcal{A}$ is a π-system if it is closed under intersection, and $\mathcal{D}$ is a Dynkin system if

- $\Omega \in \mathcal{D}$,
- $A, B \in \mathcal{D},\ A \subset B \quad \Rightarrow \quad B \setminus A \in \mathcal{D}$,

- $A_n \uparrow A$, $A_n \in \mathcal{D} \quad \Rightarrow \quad A \in \mathcal{D}$.

Clearly, any intersection of Dynkin systems is still a Dynkin system.

1. Show that $\mathcal{A}$ is a σ-algebra if and only if it is both a π-system and a Dynkin system.
2. Show that if $\mathcal{A}$ is a π-system, then $\sigma(\mathcal{A})$ is the *smallest* Dynkin system containing $\mathcal{A}$.

Exercise 4 1. Let $\Omega = \mathbb{R}$ and $\mathcal{F} = \{A \subset \Omega : A \text{ or } A^c \text{ countable}\}$. For $A \in \mathcal{F}$, define

$$\mathsf{P}(A) = \begin{cases} 1, & A^c \text{ countable}, \\ 0, & A \text{ countable}. \end{cases}$$

Show that $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space.

2. Let $\Omega = \{(x_1, x_2, \cdots) : x_i \in \{0,1\}\}$. For $n \geq 1$, define the projection $\pi_n : \Omega \to \{0,1\}^n$ by

$$\pi_n(x) = (x_1, x_2, \cdots, x_n), \quad x = (x_1, x_2, \cdots) \in \Omega.$$

Show that for each $n \geq 1$,

$$\mathcal{F}_n = \{\pi_n^{-1}(A) : A \subset \{0,1\}^n\}$$

is a σ-algebra, and $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ is an algebra. Show that $\{(0, 0, \cdots)\} \in \sigma(\mathcal{A}) \setminus \mathcal{A}$.

Remark: the last statement means that σ-algebras are not closed under union.

Exercise 5 Let μ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We say that $x \in \operatorname{supp} \mu$ (the support of μ), if $\mu(x - \varepsilon, x + \varepsilon) > 0$ for every $\varepsilon > 0$, .

1. Show that if x is a point of mass for μ, then $x \in \operatorname{supp} \mu$.
2. Show that if $\mu = \mu_X$ is the distribution of a continuous r.v. X with continuous density f, and $f(x) > 0$, then $x \in \operatorname{supp} \mu$.
3. Show that $(\operatorname{supp} \mu)^c$ is an open set, and hence $\operatorname{supp} \mu$ is a closed set.
4. Recall that the Cantor set is defined by $\mathcal{C} = [0,1] \setminus \bigcup_{n \geq 1, k \leq 2^{n-1}} I_n^{(k)}$, where

$$I_1^{(1)} = (\frac{1}{3}, \frac{2}{3}), \quad I_2^{(1)} = (\frac{1}{9}, \frac{2}{9}), \quad I_2^{(2)} = (\frac{7}{9}, \frac{8}{9}), \quad \ldots$$

and the definition of Cantor function φ (see, e.g., Example 1.2.7 in Durrett). The distribution function φ defines a probability measure $\mu = \mu_\varphi$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Show that $x \in \operatorname{supp} \mu_\varphi$ for every $x \in \partial I_n^{(k)}$, and use this to conclude that $\operatorname{supp} \mu_\varphi = \mathcal{C}$.

Hint: the Cantor set $\mathcal{C}$ is the closure of $\bigcup_{n \geq 1, k \leq 2^{n-1}} \partial I_n^{(k)}$.

Exercise 6 1. Show that a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is a measurable map from $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

2. Show that $\mathcal{B}(\mathbb{R}^d)$ is the smallest σ-algebra that makes all continuous functions measurable, i.e., if for every continuous function f, it is measurable from $(\mathbb{R}^d, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then $\mathcal{B}(\mathbb{R}^d) \subset \mathcal{F}$.