HW7

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Exercise 1 Consider the one-dimensional SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)\, dB_t$$

where b, σ satisfy:

- b is bounded, measurable, and

$$|b(t,x) - b(t,y)| \le g(|x-y|)$$

for some continuous, strictly increasing, concave function $g : \mathbb{R}_+ \to \mathbb{R}_+$ with $g(0) = 0$ and $\int_0^1 \frac{du}{g(u)} = \infty$.

- σ is bounded, measurable and

$$|\sigma(t,x) - \sigma(t,y)| \le h(|x-y|)$$

for some continuous, strictly increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $h(0) = 0$ and $\int_0^1 \frac{du}{h^2(u)} = \infty$.

Show that pathwise uniqueness holds for this SDE.

Hint: You may use the following result: for g given above, if f is a non-negative continuous function, then

$$f(t) \le \int_0^t g\big(f(s)\big)\, ds, \quad t \ge 0 \implies f(t) \equiv 0, \quad t \ge 0.$$

If $X^{(j)}$, $j = 1, 2$ are two weak solutions, you are aiming at $f(t) = \mathsf{E}|X_t^{(1)} - X_t^{(2)}|$ satisfying the above integral inequality.

Exercise 2 Let $(M_t)_{t\ge 0}$ be a c.l.m. on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbf{P})$ with

$$\langle M \rangle_t = \int_0^t a(s)\, ds$$

for some progressively measurable process $a \ge 0$. Show that there exists an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\ge0}, \tilde{\mathbf{P}})$ and a standard Brownian motion B on it such that

$$M_t = \int_0^t \sqrt{a(s)}\, dB_s.$$

Hint: consider

$$B_t := \int_0^t \mathbb{1}_{\{a(s)>0\}} \frac{1}{\sqrt{a(s)}}\, dM_s + \int_0^t \mathbb{1}_{\{a(s)=0\}}\, dW_s$$

where W is a standard Brownian motion independent of everything else.

Exercise 3 Write $\mathbf{y} = \big(y(t)\big)_{t\geq 0}$. Suppose $b(t,\mathbf{y})$ and $\sigma(t,\mathbf{y})$ are progressively measurable functionals from $[0,\infty)\times\mathcal{C}[0,\infty)$ into $\mathbb{R}$ satisfying

$$|b(t,\mathbf{y})|^2 + |\sigma(t,\mathbf{y})|^2 \leq K\Big(1+\max_{0\leq s\leq t}|y(s)|^2\Big), \quad 0\leq t<\infty,\ \mathbf{y}\in\mathcal{C}[0,\infty),$$

where K is a positive constant. Let (X,W) be a weak solution on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathsf{P})$ to the SDE

$$X_t = X_0 + \int_0^t b(s,X)\,ds + \int_0^t \sigma(s,X)\,ds,$$

and $\mathsf{E}|X_0|^{2m},\infty$ for some $m\geq 1$. Show that for any finite $T>0$,

$$\mathsf{E}\Big(\max_{0\leq s\leq t}|X_s|^{2m}\Big) \leq C\Big(1+\mathsf{E}|X_0|^{2m}\Big)e^{Ct}, \quad 0\leq t\leq T,$$

and

$$\mathsf{E}|X_t - X_s|^{2m} \leq C\big(1+\mathsf{E}|X_0|^{2m}\big)(t-s)^m, \quad 0\leq s<t\leq T,$$

where $C = C(m,T,K)$ is a constant.

Exercise 4 Suppose that $u(t,x)\in\mathcal{C}([0,t]\times\mathbb{R})\cap\mathcal{C}^{1,2}\big((0,t]\times\mathbb{R}\big)$ solves the heat equation

$$\begin{cases} \partial_t u(t,x) = \dfrac{1}{2}\partial_{xx}u(t,x) - k(t,x)u(t,x), & (t,x)\in(0,t]\times\mathbb{R}, \\ u(0,x) = f(x), & x\in\mathbb{R}, \end{cases}$$

where $k(t,x)$ is a bounded continuous function. Suppose that u satisfies the growth condition

$$\sup_{0\leq s\leq t}|u(s,x)| \leq Me^{a|x|^2}$$

for some $0<a<\frac{1}{2t}$ and $M>0$. Show that u admits the Feynman–Kac representation

$$u(t,x) = \mathsf{E}^x f(B_t)e^{-\int_0^t}k(t-s,B_s)\,ds.$$

Hint: consider $Y_s = u(t-s,B_s)e^{-\int_0^s k(t-\theta,B_\theta)\,d\theta}$; apply the Optional Sampling Theorem with respect to the stopping times $\tau_n = \inf\{s\geq 0 : |B_s|\geq n\}$ and pass to the limit carefully.