Stochastic Analysis, 2025 Spring: HW2

March 12, 2025

From Le Gall: Exercises 2.26, 2.27, 2.29, 2.33

Additional exercises:

Exercise 1 The *Brownian sheet* $(\mathbb{B}_{s,t})_{s,t\in[0,1]}$ is a centered Gaussian process with covariance

$$\mathsf{E}\mathbb{B}_{s,t}\mathbb{B}_{s',t'} = (s \wedge s')(t \wedge t'), \quad s, t, s', t' \in [0,1].$$

It can be constructed via GWN with $H = L^2([0,1]^2, \mathcal{B}([0,1]^2), ds \times dt)$ and $\mathbb{B}_{s,t} = G\big(\mathbb{1}_{[0,s]\times[0,t]}\big)$.

1. Show that for each $p \geq 1$, there is some constant $K_p > 0$,

$$\mathsf{E}|\mathbb{B}_{s,t} - \mathbb{B}_{s',t'}|^{2p} \leq K_p\big(|s-s'|^p + |t-t'|^p\big), \quad s, t, s', t' \in [0,1].$$

2. Let $0 < \gamma < 1/2$. Show that with probability one, there is a random constant $n_0 = n_0(\omega)$ such that for all $n \geq n_0$,

$$\Big|\mathbb{B}_{\frac{k}{2^n},\frac{\ell}{2^n}} - \mathbb{B}_{\frac{k'}{2^n},\frac{\ell'}{2^n}}\Big| \leq 2^{-\gamma n}, \quad 0 \leq k, \ell, k', \ell' \leq 2^n, \ |k-k'| + |\ell - \ell'| \leq 1.$$