HW9

April 17, 2024

Exercise 1 Let $(B_t)_{t\geq 0}$ be Brownian motion defined on $(\Omega, \mathcal{F}, \mathsf{P})$. Let $X_t = |B_t|$.

- Show that $(X_t)_{t\geq 0}$ is a Markov process on $\mathbb{R}_+$. Namely, for every $0 < t_1 < \cdots < t_n = t < t+s$ and $A \in \mathcal{B}(\mathbb{R}_+)$,
$$\mathsf{P}(X_{t+s} \in A \mid X_{t_1}, X_{t_2}, \ldots, X_{t_n}) = \mathsf{P}(X_{t+s} \in A \mid X_t).$$

Proof. For $t \geq 0$, let $\mathcal{F}_t := \sigma(X_s, 0 \leq s \leq t)$. Then

$P(X_{t+s} \in A \mid \mathcal{F}_t) = P(X_{t+s} \in A, B_t > 0 \mid \mathcal{F}_t) + P(X_{t+s} \in A, B_t \leq 0 \mid \mathcal{F}_t)$

$= P(X_{t+s} \in A \mid B_t > 0, \mathcal{F}_t) \cdot P(B_t > 0 \mid \mathcal{F}_t) + P(X_{t+s} \in A \mid B_t \leq 0, \mathcal{F}_t) \cdot P(B_t \leq 0 \mid \mathcal{F}_t)$

Now note that for every $0 \leq t_1 < \cdots < t_i < \cdots < t_n = t$, $a < b \in [0, +\infty)$, and $p_t(x) := \frac{1}{(2\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{2t}}$, $x \in \mathbb{R}$; By symmetry of the p_t w.r.t. 0, we have for $i \in \{1, \cdots, n\}$ that $\underline{P(X_{t_i} \in [a,b), B_t > 0) = P(B_{t_i} \in [a,b), B_t > 0)}$

$= \int_a^b p_{t_i}(x)\,dx = \int_a^b p_{t_i}(-x)\,dx = \frac{1}{2} \cdot \left(\int_a^b p_{t_i}(x)\,dx + \int_a^b p_{t_i}(-x)\,dx \right)$

$= P(B_t > 0) \cdot \left(P(B_{t_i} \in [a,b)) + P(-B_{t_i} \in [a,b)) \right)$

$= P(B_t > 0) \cdot P(|B_{t_i}| \in [a,b)) = P(B_t > 0) P(X_{t_i} \in [a,b))$. Then by monotone class Lemma, we have that $\{B_t > 0\}$ is independent of $\mathcal{F}_t$. Similarly $\{B_t \leq 0\}$ is also independent of $\mathcal{F}_t$. Thus we could continue the calculation: $P(X_{t+s} \in A \mid \mathcal{F}_t)$

$= P(X_{t+s} \in A \mid B_t > 0, \mathcal{F}_t) \cdot P(B_t > 0) + P(X_{t+s} \in A \mid B_t \leq 0, \mathcal{F}_t) \cdot P(B_t \leq 0)$

$= \frac{1}{2}\left[P(X_{t+s} \in A \mid B_t > 0, \mathcal{F}_t) + P(X_{t+s} \in A \mid B_t \leq 0, \mathcal{F}_t) \right]$. $(*)$

Note that $\{B_t > 0\} \cap \sigma(B_t) \subseteq \{B_t > 0\} \cap \mathcal{F}_t$, then we could keep going on $(*)$

$= \frac{1}{2}\left[P(X_{t+s} \in A \mid B_t > 0, B_t, \mathcal{F}_t) + P(X_{t+s} \in A \mid B_t \leq 0, B_t, \mathcal{F}_t) \right]$

$\underset{\text{M.P. of } B(\cdot)}{=} \frac{1}{2}\left[P(X_{t+s} \in A \mid B_t > 0, B_t) + P(X_{t+s} \in A \mid B_t \leq 0, B_t) \right]$

$$= \frac{1}{2}\left[P(X_{t+s} \in A \mid |B_t|, B_t > 0) + P(X_{t+s} \in A \mid |B_t|, B_t \leq 0) \right]$$

$$= \frac{1}{2}\left[\frac{P(X_{t+s} \in A, B_t > 0 \mid |B_t|)}{P(B_t > 0 \mid |B_t|)} + \frac{P(X_{t+s} \in A, B_t \leq 0 \mid |B_t|)}{P(B_t \leq 0 \mid |B_t|)} \right]$$

(By independence of $\{B_t > 0\}, \{B_t \leq 0\}$ with $\mathcal{F}_t$)

$$= \frac{1}{2}\left[\frac{P(X_{t+s} \in A, B_t > 0 \mid X_t)}{\frac{1}{2}} + \frac{P(X_{t+s} \in A, B_t \leq 0 \mid X_t)}{\frac{1}{2}} \right]$$

$$= P(X_{t+s} \in A \mid X_t).$$

$\square$

Remake: The proof of independence of $\{B_t > 0\}$ and $\mathcal{F}_t$ still has some problems. I will upload a proof as soon as I figure it out.

- Show that the Markov semi-group $(\mathsf{P}_t f)(x) = \mathsf{E}^x f(X_t)$ can be written as

$$(\mathsf{P}_t f)(x) = \int_0^\infty [g_t(x-y) + g_t(x+y)] f(y)\,dy, \quad g_t(z) = \frac{1}{\sqrt{2\pi t}} e^{-z^2/2t}.$$

proof. Under probability P^x, $|B_0| = X_0 = x$ a.s.

① If $B_0 = x$ a.s., then for $t > 0$, $B_t \sim \text{Normal}(x, t)$. Then we have

$$P_t f(x) = \mathbb{E}^x f(X_t) = \mathbb{E}^x f(|B_t|) = \mathbb{E}^x [f(B_t) 1_{\{B_t > 0\}}] + \mathbb{E}^x [f(-B_t) 1_{\{B_t \leq 0\}}]$$

$$= \int_0^{+\infty} f(y) \cdot g_t(x-y)\,dy + \int_0^{+\infty} f(y)\, g_t(x+y)\,dy$$ (Note that $\{B_t \leq 0\} = \{-B_t \geq 0\}$ and the density of $-B_t$ is $g_t(x-(-y)) = g_t(x+y)$).

$$= \int_0^{+\infty} (g_t(x-y) + g_t(x+y)) f(y)\,dy.$$

② If $B_0 = -x$ a.s., then for $t > 0$ $B_t \sim \text{Normal}(-x, t)$. Then we have

$$P_t f(x) = \mathbb{E}^x f(|B_t|) = \mathbb{E}^x [f(B_t) 1_{\{B_t > 0\}}] + \mathbb{E}^x [f(-B_t) 1_{\{B_t \leq 0\}}]$$

$$= \int_0^{+\infty} f(y) \cdot g_t(y+x)\,dy + \int_0^{+\infty} f(y) \cdot g_t(-y+x)\,dy$$

$$= \int_0^{+\infty} (g_t(x-y) + g_t(x+y)) f(y)\,dy.$$

- Denote by $\mathcal{L}$ the generator of $(\mathrm{P}_t)_{t\geq 0}$. Let $f \in \mathcal{C}_0^2(\mathbb{R}_+)$ be such that $f'(0) = 0$. Show that $f \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}f = \frac{1}{2}f''$.

  Hint: if $f'(0) = 0$, then f can be extended to an even function $g \in \mathcal{C}_0^2(\mathbb{R})$ such that $g(x) = f(|x|)$. Now apply Itô's formula to $g(B_t)$.

proof. Define $g(x) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$, then $g(x) = f(|x|)$ for all $x \in \mathbb{R}$ and $g \in C_0^2(\mathbb{R})$

By Itô's formula, $g(B_t) = g(x) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds$. Then for $x \geq 0$, $P_t f(x) = \mathbb{E}^x f(X_t) = \mathbb{E}^x g(B_t) = \mathbb{E}^x\left[g(x) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds\right]$

$= g(x) + 0 + \frac{1}{2}\mathbb{E}^x\int_0^t g''(B_s)ds = f(x) + \frac{1}{2}\mathbb{E}^x\int_0^t g''(B_s)ds$. Then we have

$$\left\|\frac{1}{t}(P_t f - f) - \frac{1}{2}f''\right\| = \sup_{x\geq 0}\left|\frac{1}{t}\cdot\frac{1}{2}\mathbb{E}^x\int_0^t g''(B_s)ds - \frac{1}{2}f''(x)\right|$$

$$= \frac{1}{2}\sup_{x\geq 0}\left|\mathbb{E}^x\left[\frac{1}{t}\int_0^t g''(B_s)ds - f''(x)\right]\right| = \frac{1}{2}\sup_{x\geq 0}\left|\mathbb{E}^x\left[\frac{1}{t}\int_0^t (g''(B_s) - f''(x))ds\right]\right|$$

$$\leq \frac{1}{2}\sup_{x\geq 0}\mathbb{E}^x\left[\frac{1}{t}\int_0^t |g''(B_s) - f''(x)|ds\right] = \frac{1}{2}\frac{1}{t}\int_0^t \sup_{x\geq 0}\mathbb{E}^x|g''(B_s) - f''(x)|ds.$$

letting $t \to 0+$, by fundamental theorem of Calculus, we have that

$$\lim_{t\to 0+}\left\|\frac{1}{t}(P_t f - f) - \frac{1}{2}f''\right\| = \lim_{t\to 0+}\frac{1}{2}\frac{1}{t}\int_0^t \sup_{x\geq 0}\mathbb{E}^x|g''(B_s) - f''(x)|ds$$

$$= \frac{1}{2}\sup_{x\geq 0}\mathbb{E}^x|g''(B_0) - f''(x)| = \frac{1}{2}\sup_{x\geq 0}\mathbb{E}^x|f''(x) - f''(x)| = 0.$$

The proof is done.

- (Optional) Show that $f \notin \mathcal{D}(\mathcal{L})$ if $f'(0) \neq 0$.

proof. For $x \geq 0$, define $g(x) := f(x) - f'(0)x$. Then we have $g \in C_0^2(\mathbb{R}_+)$ with $g'(0) = 0$. By results above, we obtain that $g \in \mathcal{D}(L)$, which means that $\lim_{t\to 0+}\frac{1}{t}(P_t g - g)$ exists in C_0 norm. We have for $x \geq 0$, $P_t g(x) - g(x) = \mathbb{E}^x g(X_t) - g(x) = \mathbb{E}^x(f(X_t) - f'(0)X_t) - g(x) = P_t f(x) - f'(0)\mathbb{E}^x X_t - g(x) = P_t f(x) - f(x) - (f'(0)\mathbb{E}^x X_t - f'(0)x)$, which implies ~~that~~

$P_t f(0) - f(0) = P_t g(0) - g(0) + f'(0)\mathbb{E}^0 X_t$. Since $\mathbb{E}^0 X_t = C\sqrt{t}$ for some constant $C > 0$, we have

$$\lim_{t\to 0+}\left(\frac{P_t f(0) - f(0)}{t}\right) = \lim_{t\to 0+}\frac{1}{t}(P_t g(0) - g(0) + f'(0)\mathbb{E}^0 X_t)$$

$$= +\infty,$$

which implies the result that $f \notin D(\mathcal{L})$ since C_0 convergence implies pointwise convergence.

Exercise 2 Give strong solutions to the following SDEs.

1. By applying Itô's formula to $\log X_t$, solve

$$dX_t = \sigma X_t\, dB_t + r X_t\, dt.$$

(Formally): $d\log X_t = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2} d\langle X\rangle_t$

$= \frac{1}{X_t}(\sigma X_t dB_t + r X_t dt) - \frac{1}{2X_t^2}\sigma^2 X_t^2 dt$

$= \sigma dB_t + r dt - \frac{1}{2}\sigma^2 dt = (r - \frac{1}{2}\sigma^2) dt + \sigma dB_t.$

$X_t = X_0 \exp\left(\int_0^t (r - \frac{1}{2}\sigma^2) ds + \int_0^t \sigma dB_t\right)$

$= X_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma B_t\right).$

It is easy to verify $(X_t)_{t\geq 0}$ is the unique strong solution to this SDE.

2. By applying Itô's formula to $e^{\lambda t} X_t$, solve

$$dX_t = dB_t - \lambda X_t\, dt.$$

(Formally): $d e^{\lambda t} X_t = \lambda e^{\lambda t} X_t dt + e^{\lambda t} dX_t$

$= \lambda e^{\lambda t} X_t dt + e^{\lambda t} dB_t - \lambda X_t e^{\lambda t} dt$

$= e^{\lambda t} dB_t.$

$e^{\lambda t} X_t - X_0 = \int_0^t e^{\lambda s} dB_s$

$\Rightarrow X_t = e^{-\lambda t}\left(X_0 + \int_0^t e^{\lambda s} dB_s\right).$

It is easy to verify $(X_t)_{t\geq 0}$ is the unique strong solution to this SDE.

3. By applying Itô's formula to $\frac{X_t}{T-t}$, solve

$$dX_t = -\frac{X_t}{T-t}\,dt + dB_t, \quad 0 \le t < T, \qquad X_0 = 0.$$

(Formally): $d\left(\frac{X_t}{T-t}\right) = \frac{X_t}{(T-t)^2}dt + \frac{1}{T-t}dX_t$

$$= \frac{X_t}{(T-t)^2}dt + \frac{1}{T-t}\left(-\frac{X_t}{T-t}dt + dB_t\right)$$

$$= \frac{1}{T-t}dB_t$$

$$\Longrightarrow \frac{X_t}{T-t} - \frac{X_0}{T} = \int_0^t \frac{1}{T-s}\,dB_s .$$

$$\Longrightarrow X_t = (T-t)\int_0^t \frac{1}{T-s}\,dB_s .$$

It is easy to verify $(X_t)_{t\geq 0}$ is the unique strong solution to this SDE.