HW9

April 17, 2024

Exercise 1 Let $(B_t)_{t\geq 0}$ be Brownian motion defined on $(\Omega, \mathcal{F}, \mathsf{P})$. Let $X_t = |B_t|$.

- Show that $(X_t)_{t\geq 0}$ is a Markov process on $\mathbb{R}_+$. Namely, for every $0 < t_1 < \cdots < t_n = t < t+s$ and $A \in \mathcal{B}(\mathbb{R}_+)$,
$$\mathsf{P}(X_{t+s} \in A \mid X_{t_1}, X_{t_2}, \ldots, X_{t_n}) = \mathsf{P}(X_{t+s} \in A \mid X_t).$$

- Show that the Markov semi-group $(\mathsf{P}_t f)(x) = \mathsf{E}^x f(X_t)$ can be written as
$$(\mathsf{P}_t f)(x) = \int_0^\infty \big[g_t(x-y) + g_t(x+y)\big] f(y)\, dy, \quad g_t(z) = \frac{1}{\sqrt{2\pi t}} e^{-z^2/2t}.$$

- Denote by $\mathcal{L}$ the generator of $(\mathsf{P}_t)_{t\geq 0}$. Let $f \in \mathcal{C}_0^2(\mathbb{R}_+)$ be such that $f'(0) = 0$. Show that $f \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}f = \frac{1}{2} f''$.

  Hint: if $f'(0) = 0$, then f can be extended to an even function $g \in \mathcal{C}_0^2(\mathbb{R})$ such that $g(x) = f(|x|)$. Now apply Itô's formula to $g(B_t)$.

- (Optional) Show that $f \notin \mathcal{D}(\mathcal{L})$ if $f'(0) \neq 0$.

Exercise 2 Give strong solutions to the following SDEs.

1. By applying Itô's formula to $\log X_t$, solve
$$dX_t = \sigma X_t\, dB_t + r X_t\, dt.$$

2. By applying Itô's formula to $e^{\lambda t} X_t$, solve
$$dX_t = dB_t - \lambda X_t\, dt.$$

3. By applying Itô's formula to $\frac{X_t}{T-t}$, solve
$$dX_t = -\frac{X_t}{T-t}\, dt + dB_t, \quad 0 \leq t < T, \qquad X_0 = 0.$$