HW8

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Exercise 1 (KS, Ex 3.5.18) Let $B = (B_t)_{0 \le t \le 1}$ be a Brownian motion. Define

$$T = \inf\{0 \le t \le 1 : t + B_t^2 = 1\}$$

and

$$X_t = -\frac{2}{(1-t)^2} B_t \mathbb{1}_{\{t \le T, t<1\}}, \quad 0 \le t < 1.$$

1. Show that $\mathsf{P}(T < 1) = 1$, and hence $\int_0^1 X_t^2 \, dt < \infty$ a.s.

proof. By definition of T, we know $0 \le T(\omega) \le 1$ for $\omega \in \Omega$. Suppose there exists a subset $A \subseteq \Omega$ with $P(A) > 0$ such that $T(\omega) = 1$ for $\omega \in A$, then by continuity of $B(\cdot, \omega)$ we have for $\omega \in A$. $1 \geq T(\omega) + B^2_{T(\omega)}(\omega) = 1 + B_1^2(\omega)$, which means that $B_1(\omega) = 0$ for $\omega \in A$. It contradicts the fact that B_1 follows the distribution $Normal(0,1)$ (which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$). Therefore $P(T<1) = 1$.

Then for a.s. ω, $T(\omega) < 1$, which implies that:

$|X_t|^2 = \left|-\frac{2}{(1-t)^2} B_t 1_{\{t \le T, t<1\}}\right|^2 = \frac{4}{(1-t)^4} B_t^2(\omega) 1_{\{t \le T, t<1\}} = \frac{4}{(1-t)^4} B_t^2(\omega) 1_{\{t \le T\}}(t,\omega)$ is bounded in $t \in [0,1)$ for a.s. $\omega \in \Omega$, which means that $\int_0^1 X_t^2 \, dt < \infty$ for a.s. $\omega \in \Omega$.

2. Apply Itô's formula to the process $(1-t)^{-2}B_t^2$ to conclude that

$$\int_0^1 X_t\,dB_t - \frac{1}{2}\int_0^1 X_t^2\,dt = -1 - 2\int_0^T \left[\frac{1}{(1-t)^4} - \frac{1}{(1-t)^3}\right] B_t^2\,dt \le -1.$$

proof. By Itô's formula, for $t \in [0,1)$, $(1-t)^{-2}B_t^2 = 0 + \int_0^t 2(1-s)^{-3}B_s^2 ds$ $+ \int_0^t 2(1-s)^{-2}B_s\,dB_s + \int_0^t (1-s)^{-2}ds$, which means that:

$$-\int_0^t 2(1-s)^{-2}B_s\,dB_s = -(1-t)^{-2}B_t^2 + \int_0^t \frac{2}{(1-s)^3}B_s^2\,ds + \int_0^t \frac{1}{(1-s)^2}ds.$$

Then note that T is a stopping time and by properties of Stochastic integral, we have that:

$$\int_0^1 X_t\,dB_t = \int_0^1 -\frac{2}{(1-t)^2}B_t 1_{\{t\le T\}}dB_t = \int_0^T -\frac{2}{(1-t)^2}B_t\,dB_t = -\frac{1}{(1-T)^2}B_T^2 +$$
$$\int_0^T \frac{2}{(1-t)^3}B_t^2\,dt + \int_0^T \frac{1}{(1-t)^2}dt = -\frac{1}{1-T} + \int_0^T \frac{2}{(1-t)^3}B_t^2\,dt + \int_0^T \frac{1}{(1-t)^2}dt$$
$$= -\frac{1}{1-T} + \int_0^T \frac{2}{(1-t)^3}B_t^2\,dt + \frac{1}{1-T} - 1.$$

And $\frac{1}{2}\int_0^1 X_t^2\,dt = \frac{1}{2}\int_0^1 \frac{4}{(1-t)^4}B_t^2 1_{\{t\le T\}}dt = \int_0^T \frac{2}{(1-t)^4}B_t^2\,dt$.

Thus $\int_0^1 X_t\,dB_t - \frac{1}{2}\int_0^1 X_t^2\,dt = -1 - 2\int_0^T \left[\frac{1}{(1-t)^4} - \frac{1}{(1-t)^3}\right] B_t^2\,dt$

≤ -1. (Since $\frac{1}{(1-t)^4} - \frac{1}{(1-t)^3} \ge 0$ for $t \in [0,1)$)

3. Show that the exponential super-martingale $Z_t(X)$, $0 \le t \le 1$ is not a martingale; however, for each $n \ge 1$ and $\sigma_n = 1 - (1/\sqrt{n})$, $Z_{t\wedge\sigma_n}(X)$, $0 \le t \le 1$ is a martingale.

proof. $Z_t(X) = \exp\left(\int_0^t X_s\,dB_s - \frac{1}{2}\int_0^t X_s^2\,ds\right)$. Obviously we have that $Z_0(X) = 1$, which implies that $\mathbb{E}Z_1(X) = 1$. But by 2.) we obtain that $Z_1(X) \le \exp(-1)$, which means that

$\mathbb{E}Z_0(X) = 1 > \exp(-1) \ge \mathbb{E}Z_1(X)$, which implies that $(Z_t(X))_{t\in[0,1)}$ is not a martingale.

To show that $(Z_{t\wedge\sigma_n})_{t\in[0,1]}$ is a martingale for each $n\geq 1$, by Novikov's criterion it suffices to show $\mathbb{E}[\exp(\frac{1}{2}\int_0^{\sigma_n} X_t^2 dt)] < \infty$. We have

$$\mathbb{E}[\exp(\tfrac{1}{2}\int_0^{\sigma_n} X_t^2 dt)] = \mathbb{E}[\exp(\tfrac{1}{2}\int_0^{\sigma_n} \frac{4}{(1-t)^4} B_t^2 1_{\{t\leq T\}} dt)]$$

$$\leq \mathbb{E}[\exp(2\int_0^{\sigma_n} \frac{1}{(1-t)^3} dt)] = \exp(2\int_0^{\sigma_n} \frac{1}{(1-t)^3} dt) < \infty.$$

The proof is therefore done.

Exercise 2 (Le Gall, Ex 5.28) Let B be a Brownian motion **started from** 1. Fix $\varepsilon \in (0,1)$ and set $T_\varepsilon = \inf\{t \geq 0 : B_t = \varepsilon\}$. Also let $\lambda > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$.

1. Show that $Z_t = (B_{t\wedge T_\varepsilon})^\alpha$ is a semi-martingale and give its canonical decomposition as the sum of a c.l.m. and a finite variation process.

Proof. For $t \geq 0$, $B_{t\wedge T_\varepsilon} = \int_0^{t\wedge T_\varepsilon} 1\, dB_s = \int_0^t 1_{\{s\leq T_\varepsilon\}} dB_s$.

Then by Itô's formula,

$$Z_t = (B_{t\wedge T_\varepsilon})^\alpha = 1 + \int_0^t \alpha (B_{s\wedge T_\varepsilon})^{\alpha-1} 1_{\{s\leq T_\varepsilon\}} dB_s$$

$$+ \int_0^t \tfrac{1}{2}\alpha(\alpha-1)(B_{s\wedge T_\varepsilon})^{\alpha-2} 1_{\{s\leq T_\varepsilon\}} ds.$$

Set for $t\geq 0$ that

$M_t := \int_0^t \alpha (B_{s\wedge T_\varepsilon})^{\alpha-1} 1_{\{s\leq T_\varepsilon\}} dB_s$, which is a c.l.m. and

$A_t := \int_0^t \frac{1}{2}\alpha(\alpha-1)(B_{s\wedge T_\varepsilon})^{\alpha-2} 1_{\{s\leq T_\varepsilon\}} ds$, which is a f.v. process.

2. Show that the process

$$Z_t = (B_{t\wedge T_\varepsilon})^\alpha \exp\Big(-\lambda \int_0^{t\wedge T_\varepsilon} \frac{ds}{B_s^2}\Big)$$

is a c.l.m. if α and λ satisfy a polynomial equation to be determined.

proof. For $t \geqslant 0$, $Z_t = (B_{t\wedge T_\varepsilon})^\alpha \exp\Big(-\lambda \int_0^t \frac{1_{\{s\leqslant T_\varepsilon\}}}{B_s^2} ds\Big)$

$$= 1 + \int_0^t (B_{s\wedge T_\varepsilon})^\alpha \exp\Big(-\lambda \int_0^s \frac{1_{\{\tau\leqslant T_\varepsilon\}}}{B_\tau^2} d\tau\Big) \cdot \Big(-\lambda 1_{\{s\leqslant T_\varepsilon\}} \cdot \frac{1}{B_s^2}\Big) ds$$

$$+ \int_0^t \exp\Big(\lambda \int_0^s \frac{1_{\{\tau\leqslant T_\varepsilon\}}}{B_\tau^2} d\tau\Big) \cdot \alpha (B_{s\wedge T_\varepsilon})^{\alpha-1} 1_{\{s\leqslant T_\varepsilon\}} dB_s$$

$$+ \int_0^t \exp\Big(\lambda \int_0^s \frac{1_{\{\tau\leqslant T_\varepsilon\}}}{B_\tau^2} d\tau\Big) \frac{1}{2}\alpha(\alpha-1)(B_{s\wedge T_\varepsilon})^{\alpha-2} 1_{\{s\leqslant T_\varepsilon\}} ds.$$

To make $(Z_t)_{t\geqslant 0}$ a c.l.m., it's equivalent to have: For $s \geqslant 0$,

$$(B_{s\wedge T_\varepsilon})^\alpha (-\lambda) 1_{\{s\leqslant T_\varepsilon\}} \cdot \frac{1}{B_{s\wedge T_\varepsilon}^2} + \frac{1}{2}\alpha(\alpha-1)(B_{s\wedge T_\varepsilon})^{\alpha-2} 1_{\{s\leqslant T_\varepsilon\}} = 0,$$

which means that $\lambda = \frac{1}{2}\alpha(\alpha-1)$.

3. Compute

$$\mathsf{E}\Big[\exp\big(-\lambda\int_0^{T_\varepsilon}\frac{ds}{B_s^2}\big)\Big].$$

proof. Let Z_t be the same as in 2.). And let $\alpha \in \mathbb{R}\setminus\{0\}$ such that $\lambda = \frac{1}{2}\alpha(\alpha-1)$. (It always exists such α since $\lambda>0$). There exists a localizing sequence $(S_n)_{n\geq 1}$ such that for each $n \geq 1$, $(Z_{t\wedge S_n})_{t\geq 0}$ is a uniformly integrable martingale. Since T_ε is a stopping time, by O.S.T. we have $1 = \mathbb{E}Z_0 = \mathbb{E}Z_{0\wedge S_n} = \mathbb{E}Z_{T_\varepsilon\wedge S_n} = \mathbb{E}\Big[(B_{T_\varepsilon\wedge S_n})^\alpha \exp\big(-\lambda\int_0^{T_\varepsilon\wedge S_n}\frac{1}{B_s^2}ds\big)\Big]$. Note that

$$\lim_{n\to+\infty}(B_{T_\varepsilon\wedge S_n})^\alpha \exp\big(-\lambda\int_0^{T_\varepsilon\wedge S_n}\frac{1}{B_s^2}ds\big) = B_{T_\varepsilon}^\alpha \exp\big(-\lambda\int_0^{T_\varepsilon}\frac{1}{B_s^2}ds\big),$$

and for each $n\geq 1$, $\Big|(B_{T_\varepsilon\wedge S_n})^\alpha \exp\big(-\lambda\int_0^{T_\varepsilon\wedge S_n}\frac{ds}{B_s^2}\big)\Big| \leq (B_{T_\varepsilon\wedge S_n})^\alpha$.

By the equation $\lambda = \frac{1}{2}\alpha(\alpha-1)$, we have $\alpha = \frac{1\pm\sqrt{1+8\lambda}}{2}$.

If we take $\alpha = \frac{1-\sqrt{1+8\lambda}}{2} < 0$, then we have $(B_{T_\varepsilon\wedge S_n})^\alpha \leq B_{T_\varepsilon}^\alpha = \varepsilon^\alpha$ and by D.C.T. we have that $1 = \mathbb{E}\Big[B_{T_\varepsilon}^\alpha \exp\big(-\lambda\int_0^{T_\varepsilon}\frac{ds}{B_s^2}\big)\Big] = \varepsilon^\alpha\,\mathbb{E}\Big[\exp\big(-\lambda\int_0^{T_\varepsilon}\frac{1}{B_s^2}ds\big)\Big]$, which implies that

$$\mathbb{E}\Big[\exp\big(-\lambda\int_0^{T_\varepsilon}\frac{ds}{B_s^2}\big)\Big] = \varepsilon^{-\alpha}, \quad \text{where } \alpha = \frac{1-\sqrt{1+8\lambda}}{2}.$$