HW8

April 10, 2024

Exercise 1 (KS, Ex 3.5.18) Let $B = (B_t)_{0 \le t \le 1}$ be a Brownian motion. Define

$$T = \inf\{0 \le t \le 1 : t + B_t^2 = 1\}$$

and

$$X_t = -\frac{2}{(1-t)^2} B_t \mathbb{1}_{\{t \le T, t < 1\}}, \quad 0 \le t < 1.$$

1. Show that $\mathsf{P}(T < 1) = 1$, and hence $\int_0^1 X_t^2 \, dt < \infty$ a.s.
2. Apply Itô's formula to the process $(1-t)^{-2} B_t^2$ to conclude that
$$\int_0^1 X_t \, dB_t - \frac{1}{2} \int_0^1 X_t^2 \, dt = -1 - 2 \int_0^T \Big[\frac{1}{(1-t)^4} - \frac{1}{(1-t)^3} \Big] B_t^2 \, dt \le -1.$$
3. Show that the exponential super-martingale $Z_t(X)$, $0 \le t \le 1$ is not a martingale; however, for each $n \ge 1$ and $\sigma_n = 1 - (1/\sqrt{n})$, $Z_{t \wedge \sigma_n}(X)$, $0 \le t \le 1$ is a martingale.

Exercise 2 (Le Gall, Ex 5.28) Let B be a Brownian motion **started from** 1. Fix $\varepsilon \in (0,1)$ and set $T_\varepsilon = \inf\{t \ge 0 : B_t = \varepsilon\}$. Also let $\lambda > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$.

1. Show that $Z_t = (B_{t \wedge T_\varepsilon})^\alpha$ is a semi-martingale and give its canonical decomposition as the sum of a c.l.m. and a finite variation process.
2. Show that the process
$$Z_t = (B_{t \wedge T_\varepsilon})^\alpha \exp\Big(-\lambda \int_0^{t \wedge T_\varepsilon} \frac{ds}{B_s^2} \Big)$$
is a c.l.m. if α and λ satisfy a polynomial equation to be determined.
3. Compute
$$\mathsf{E}\Big[\exp\big(-\lambda \int_0^{T_\varepsilon} \frac{ds}{B_s^2} \big) \Big].$$