HW 7

Ex 1.

1. By definition of Stratonovich integral and properties of Itô's integral, we have that

$$\langle Z_\cdot, \int_0^\cdot Y_s \circ dX_s \rangle_t = \langle Z_\cdot, \int_0^\cdot Y_s \, dX_s + \frac{1}{2}\langle X, Y\rangle_\cdot \rangle_t$$

$$= \langle Z_\cdot, \int_0^\cdot Y_s \, dX_s \rangle_t + 0 = \int_0^t Y_s \, d\langle X_\cdot, Z\rangle_s$$

2. $$\int_0^t Z_s Y_s \circ dX_s = \int_0^t Z_s Y_s \, dX_s + \frac{1}{2}\langle ZY, X\rangle_t$$

(Itô's formula)

$$= \int_0^t Z_s Y_s \, dX_s + \frac{1}{2}\langle \int_0^\cdot Z_s \, dY_s + \int_0^\cdot Y_s \, dZ_s + \int_0^\cdot d\langle Z, Y\rangle$$

$$, X\rangle_t$$

$$= \int_0^t Z_s Y_s \, dX_s + \frac{1}{2}\Big(\int_0^t Z_s \, d\langle X, Y\rangle_s + \int_0^t Y_s \, d\langle X, Z\rangle_s$$

$$+ 0\Big)$$

$$= \int_0^t Z_s Y_s \, dX_s + \frac{1}{2}\Big(\int_0^t Z_s \, d\langle Y, X\rangle_s + \int_0^t Y_s \, d\langle Z, X\rangle_s$$

$$+ \langle Z_\cdot, \frac{1}{2}\langle Y, X\rangle_\cdot \rangle_t\Big)$$

$$= \int_0^t Z_s \, d\left(\int_0^s Y_r \, dX_r + \frac{1}{2}\langle Y, X\rangle_s\right) + \frac{1}{2}\langle Z, \int_0^\cdot Y_r \, dX_r + \frac{1}{2}\langle Y, X\rangle_\cdot \rangle_t$$

$$= \int_0^t Z_s \, d\int_0^s Y_r \circ dX_r + \frac{1}{2}\langle Z, \int_0^\cdot Y_r \circ dX_r\rangle_t$$

$$= \int_0^t Z_s \circ d\left(\int_0^s Y_r \circ dX_r\right).$$

Ex 2.

1、 $T(\omega) := \inf\{t \geq 0, W_t^{(1)} = W_t^{(2)}\} = \inf\{t \geq 0, W_t^{(1)} - W_t^{(2)} = 0\}$.

Since $(W_t^{(1)} - W_t^{(2)})_{t \geq 0}$ is progressive and continuous, and $\{0\} \subseteq \mathbb{R}$ is closed, thus T is a stopping time. (By Prop 3.3 of the Lecture note).

2. Define for $t \geq 0$ $Y_t^{(1)} = Z_t^{(2)} := \mathbb{1}_{\{[0, T(\omega))\}}(t,\omega) + \frac{1}{\sqrt{2}} \mathbb{1}_{\{[T(\omega), +\infty)\}}(t,\omega)$

$Z_t^{(1)} := \frac{1}{\sqrt{2}} \mathbb{1}_{\{[T(\omega), +\infty)\}}(t,\omega) =: Y_t^{(2)}$.

Then obviously these are bounded process. And obviously they are ~~st~~ adapted to $(\mathcal{F}_t)_{t \geq 0}$ generated by $(W_t^{(1)}, W_t^{(2)})_{t \geq 0}$ since T is a stopping time. By right continuity of the

sample paths, we obtain that $Y^{(i)}, Z^{(i)}$ $(i=1,2)$ are progressive.
And then it is easily verified that for $i=1,2$, $t\geq 0$

$$B_t^{(i)} = x_i + \int_0^t Y_s^{(i)} dW_s^{(1)} + \int_0^t Z_s^{(i)} dW_s^{(2)}.$$

3. By 2.) we know that $(B_t^{(i)})_{t\geq 0}$ is a c.l.m., $i=1,2$.
And since $\langle B_\cdot^{(i)} \rangle_t = \langle \int_0^\cdot Y_s^{(i)} dW_s^{(1)} \rangle_t + \langle \int_0^\cdot Z_s^{(i)} dW_s^{(2)} \rangle_t$
$= \int_0^t Y_s^{(i)^2} ds + \int_0^t Z_s^{(i)^2} ds = \int_0^t 1_{[0,T)} + \frac{1}{2} \cdot 1_{[T,+\infty)} + \frac{1}{2} \cdot 1_{[T,+\infty)}$
$ds = \int_0^t 1\, ds = t$, by Levy's characterization, we know
that $(B_t^{(i)})_{t\geq 0}$ is B.M. for $i=1,2$.

4. $\langle B^{(1)}, B^{(2)} \rangle_t = \langle \int_0^\cdot Y_s^{(1)} dW_s^{(1)}, \int_0^\cdot Y_s^{(2)} dW_s^{(1)} \rangle_t +$
$\langle \int_0^\cdot Z_s^{(1)} dW_s^{(2)}, \int_0^\cdot Z_s^{(2)} dW_s^{(2)} \rangle_t = \int_0^t Y_s^{(1)} Y_s^{(2)} ds +$
$\int_0^t Z_s^{(1)} Z_s^{(2)} ds = \int_0^t 2 \cdot (\frac{1}{\sqrt{2}})^2 1_{[T,+\infty)} ds = \int_0^t 1_{[T,+\infty)} ds$
$= (t-T) 1_{\{T\leq t\}} = (t-T) \vee 0$.

Ex 3.

1. For $i=1,2$, continuity of $(B_t^{(i)})_{t\geq 0}$ is obtained from continuity of $W=(W_t^{(1)}, W_t^{(2)})_{t\geq 0}$ and u, v. Adaptedness is obtained by adaptedness of W and measurability of u, v. For $t\geq 0$, $\mathbb{E}|B_t^{(1)}| = \mathbb{E}|W_t^{(1)^2} - W_t^{(2)^2}| \leq \mathbb{E}|W_t^{(1)}|^2 + \mathbb{E}|W_t^{(2)}|^2 = 2t < \infty$. $\mathbb{E}|B_t^{(2)}| = \mathbb{E}|2W_t^{(1)}W_t^{(2)}|$

$= 2\,\mathbb{E}\left[|W_t^{(1)}|\cdot|W_t^{(2)}|\right] \leq 2\left(\mathbb{E}|W_t^{(1)}|^2\,\mathbb{E}|W_t^{(2)}|^2\right)^{\frac{1}{2}} = 2t < \infty$.

Integrability is done. And for $0\leq s<t<+\infty$,

$\mathbb{E}[B_t^{(1)}|\mathcal{F}_s] = \mathbb{E}[W_t^{(1)^2} - W_t^{(2)^2}|\mathcal{F}_s] = B_s^{(1)} + \mathbb{E}\Big[W_t^{(1)^2} - W_t^{(2)^2}$
$- (W_s^{(1)^2} - W_s^{(2)^2})|\mathcal{F}_s\Big] = \mathbb{E}\Big[\sum_{i=1,2}(-1)^{i-1}(W_t^{(i)} - W_s^{(i)})(W_t^{(i)} + W_s^{(i)})|\mathcal{F}_s\Big]$
$+ B_s^{(1)} = \mathbb{E}\Big[\sum_{i=1,2}(-1)^{i-1}(W_t^{(i)} - W_s^{(i)})(W_t^{(i)} - W_s^{(i)} + 2W_s^{(i)})|\mathcal{F}_s\Big]$
$+ B_s^{(1)} = \sum_{i=1,2}(-1)^{i-1}\Big(\mathbb{E}\big[(W_t^{(i)} - W_s^{(i)})^2\big] + 2W_s^{(i)}\,\mathbb{E}\big[W_t^{(i)} - W_s^{(i)}\big]\Big)$
$+ B_s^{(1)} = (t-s) - (t-s) + B_s^{(1)} = B_s^{(1)}$. $\mathbb{E}[B_t^{(2)}|\mathcal{F}_s]$

$= \mathbb{E}[2W_t^{(1)}W_t^{(2)}|\mathcal{F}_s] = 2W_s^{(1)}W_s^{(2)} + 2\,\mathbb{E}\Big[W_t^{(1)}W_t^{(2)} - W_s^{(1)}W_s^{(2)}|\mathcal{F}_s\Big]$

$= B_s^{(2)} + 2\,\mathbb{E}\Big[W_t^{(1)}W_t^{(2)} - W_s^{(1)}W_t^{(2)} + W_s^{(1)}W_t^{(2)} - W_s^{(1)}W_s^{(2)}|\mathcal{F}_s\Big]$

$= B_s^{(2)} + 2\,\mathbb{E}\Big[(W^{(1)} - W_s^{(1)})(W_t^{(2)} - W_s^{(2)} + W_s^{(2)})|\mathcal{F}_s\Big] + 0$

$= B_s^{(2)} + 2\,\mathbb{E}\left[(W_t^{(1)} - W_s^{(1)})(W_t^{(2)} - W_s^{(2)}) \mid \mathcal{F}_s\right] + 0$

$= B_s^{(2)} + 2\,\mathbb{E}\left[W_t^{(1)} - W_s^{(1)}\right]\mathbb{E}\left[W_t^{(2)} - W_t^{(2)}\right] = B_t^{(2)}$.

Thus $(B_t^{(i)})_{t\geq 0}$ is a continuous martingale for $i=1,2$.

And by Itô's formula, $B_t^{(1)} = {W_t^{(1)}}^2 - {W_t^{(2)}}^2 =$

$\sum_{i=1,2} (-1)^{i-1}\left(2\int_0^t W_s^{(i)} dW_s^{(i)} + \int_0^t 1\, ds\right) = 2\left(\int_0^t W_s^{(1)} dW_s^{(1)} - \int_0^t W_s^{(2)} dW_s^{(2)}\right)$

Then $\langle B^{(1)}\rangle_t = 4\cdot\left(\langle \int_0^\cdot W_s^{(1)} dW_s^{(1)}\rangle_t + \langle \int_0^\cdot W_s^{(2)} dW_s^{(2)}\rangle_t + 0\right)$

$= 4\cdot\left(\int_0^t {W_s^{(1)}}^2 ds + \int_0^t {W_s^{(2)}}^2 ds\right)$. On the other hand,

$B_t^{(2)} = 2W_t^{(1)} W_t^{(2)} = 2\left(\int_0^t W_s^{(2)} dW_s^{(1)} + \int_0^t W_s^{(1)} dW_s^{(2)} + \int_0^t 1\, d0\right)$,

which means $\langle B^{(2)}\rangle_t = 4\cdot\left(\int_0^t {W_s^{(2)}}^2 ds + \int_0^t {W_s^{(1)}}^2 ds\right)$.

Thus we obtain that $\langle B^{(1)}\rangle_t = \langle B^{(2)}\rangle_t$ for $t \geq 0$.

And $\langle B^{(1)}, B^{(2)}\rangle_t = 4\cdot\left(\int_0^t W_s^{(1)} W_s^{(2)} ds - \int_0^t W_s^{(2)} W_s^{(1)} ds\right) = 0$.

2.

Define $\varphi: [0,+\infty)\times\Omega \mapsto [0,+\infty)$, $(t,\omega) \mapsto \langle B^{(1)}\rangle_t(\omega) = \langle B^{(2)}\rangle_t(\omega)$.

Obviously φ is strictly increasing (by 1.)) and continuous.

Define for $r \geq 0$ $\tau_r := \inf\{t \geq 0, \langle B^{(1)}\rangle_t \geq r\}$. Then $(B_{\tau_r}^{(1)})_{r\geq 0}$ is a standard B.M. ~~By 1.) we~~ And since $\langle B^{(1)}\rangle_t = \langle B^{(2)}\rangle_t$ for $t \geq 0$, we know that $(B_{\tau_r}^{(2)})_{r\geq 0}$ is also a standard B.M..

By Dambis-Dubins-Schwarz Theorem, for $t \geq 0$

$$(B_t^{(1)}, B_t^{(2)}) = (B^{(1)}_{\tau_{\varphi(t)}}, B^{(2)}_{\tau_{\varphi(t)}}) \text{ a.s.}$$

To finish the proof, it suffices to show that (by Levy's characterization) $\langle B^{(1)}_{\tau_\cdot}, B^{(2)}_{\tau_\cdot} \rangle_t = 0$ for $t \geq 0$.

But we have $\langle B^{(1)}_\cdot, B^{(2)}_\cdot \rangle_t = 0$ for $t \geq 0$, which means that $(B_t^{(1)} B_t^{(2)})_{t \geq 0}$ is a continuous martingale. By O.S.T., we obtain that $(B^{(1)}_{\tau_t \wedge n} B^{(2)}_{\tau_t \wedge n})_{t \geq 0}$ is a martingale, which implies that $\langle B^{(1)}_{\tau_\cdot \wedge n}, B^{(2)}_{\tau_\cdot \wedge n} \rangle_t = 0$ for each $t \geq 0$, $n \in \mathbb{N}^*$. Then the proof is done by letting $n \to +\infty$ using the approximation argument of the cross-variation.

3. It is a direct consequence of 2.) since the probabilities $P(T_+^B > T_-^B)$ and $P(T_+^W > T_-^W)$ are completely determined by distributions of $(B_t)_{t \geq 0}$ and $(W_t)_{t \geq 0}$. Since $(\tilde{B}_t)$ and $(W_t)_{t \geq 0}$ have the same distribution on $(\mathbb{R}^2)^{[0,+\infty)}$, then the proof is done by results of 2.).