HW7

April 3, 2024

Exercise 1 For two c.l.sm.'s X and Y, define the *Stratonovich integral*

$$\int_0^t Y_s \circ dX_s := \int_0^t Y_s \, dX_s + \frac{1}{2}\langle X, Y\rangle_t.$$

1. Show that if Z is another c.l.sm., then

$$\langle Z, \int Y \circ dX\rangle_t = \int_0^t Y_s \, d\langle X, Z\rangle_s.$$

2. Verify the *chain rule* for Stratonovich integrals: for c.l.sm.'s X, Y and Z,

$$\int_0^t Z_s Y_s \circ dX_s = \int_0^t Z_s \circ d\Big(\int_0^s Y_r \circ dX_r\Big).$$

Exercise 2 Let (W_t^1) and (W_t^2) be two independent Brownian motions starts at $W_0^i = x_i$, $x_1 \neq x_2$. Let $T = T(\omega) = \inf\{t \geq 0 : W_t^1 = W_t^2\}$ be their collision time. Define

$$B_t^i(\omega) = \begin{cases} W_t^i, & t < T(\omega), \\ W_T^i + \dfrac{1}{\sqrt{2}}(W_t^1 + W_t^2 - W_T^1 - W_T^2), & t \geq T(\omega), \end{cases} \qquad i = 1, 2.$$

1. Explain why $T(\omega)$ is a stopping time.
2. Find bounded progressively measurable processes (Y_t^i), (Z_t^i), $i = 1, 2$, such that

$$B_t^i = x_i + \int_0^t Y_s^i \, dW_s^1 + \int_0^t Z_s^i \, dW_s^2, \quad i = 1, 2.$$

3. Use Lévy's characterization to show that (B_t^1) and (B_t^2) are Brownian motions (starting at $x_{1,2}$).
4. Show that $\langle B^1, B^2\rangle_t = (t - T) \vee 0$.

Exercise 3 Let $W = (W^1, W^2)$ be a standard two-dimensional Brownian motion starting from the origin. For functions

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy,$$

define

$$B_t^1 = u(W_t^1, W_t^2), \quad B_t^2 = v(W_t^1, W_t^2).$$

1. Show that B_t^i, $i = 1, 2$, are continuous martingales with

$$\langle B^1\rangle_t = \langle B^2\rangle_t, \quad \langle B^1, B^2\rangle_t = 0,$$

2. Show that there exists an random strictly increasing continuous function φ such that

$$(B_t^1, B_t^2) = (\tilde{B}^1_{\varphi(t)}, \tilde{B}^2_{\varphi(t)}),$$

such that $\tilde{B} = (\tilde{B}^1, \tilde{B}^2)$ is a standard two-dimensional Brownian motion.

3. For a two-dimensional continuous process $X = (X_t^1, X_t^2)$, define

$$T_\pm^X = \inf\{t \geq 0 : (X_t^1 \pm 10)^2 + (X_t^2)^2 \leq 1\}$$

to be the hitting times of the unit disks centered at $(\pm 10, 0)$ of X. Show that

$$\mathsf{P}(T_+^B > T_-^B) = \mathsf{P}(T_+^W > T_-^W).$$