Ex 5.32

1. Since for $\varepsilon > 0$, $g_\varepsilon(x) = \sqrt{\varepsilon + x^2}$ is $C^2(\mathbb{R})$. Then by Itô's formula, (*) $g_\varepsilon(B_t) = g_\varepsilon(B_0) + \int_0^t \frac{B_s}{\sqrt{\varepsilon + B_s^2}} dB_s + \frac{1}{2} \int_0^t \frac{\varepsilon}{(\varepsilon + B_s^2)^{\frac{3}{2}}} dt$ for $t \geq 0$. Then let $M_t^\varepsilon := \int_0^t \frac{B_s}{\sqrt{\varepsilon + B_s^2}} dB_s$ and $A_t^\varepsilon := \frac{1}{2} \varepsilon \int_0^t \frac{1}{\sqrt{\varepsilon + B_s^2}} ds$, $t \geq 0$, $\varepsilon > 0$. Then by properties of stochastic integral, $M^\varepsilon(\cdot)$ obviously satisfies the requirements of this question. And since $\frac{\varepsilon}{\sqrt{\varepsilon + B_s^2}} \geq 0$ for $s \geq 0$, then $A^\varepsilon(\cdot)$ is an increasing process.

2. For each $t \geq 0$, if $B_t = 0$, then $\frac{B_t}{\sqrt{\varepsilon + B_t^2}} = 0 = sgn(B_t)$ for all $\varepsilon > 0$; if $B_t \neq 0$, then $\lim_{\varepsilon \to 0+} \frac{B_t}{\sqrt{\varepsilon + B_t^2}} = \frac{B_t}{|B_t|} = sgn(B_t)$.

Thus we have that $\left| \frac{B_t}{\sqrt{\varepsilon + B_t^2}} - sgn(B_t) \right| \xrightarrow{\varepsilon \to 0+} 0$ for all ω.

Then

$$\mathbb{E} \left| \int_0^t \frac{B_s}{\sqrt{\varepsilon + B_s^2}} dB_s - \int_0^t sgn(B_s) dB_s \right|^2$$

$$= \mathbb{E} \left| \int_0^t \left(\frac{B_s}{\sqrt{\varepsilon + B_s^2}} - sgn(B_s) \right) dB_s \right|^2$$

$$= \mathbb{E} \left[\int_0^t \left| \frac{B_s}{\sqrt{\varepsilon + B_s^2}} - sgn(B_s) \right|^2 ds \right] \xrightarrow{\varepsilon \to 0+} 0 \text{ by D.C.T.}$$

$$\left(\text{since } \left| \frac{B_s}{\sqrt{\varepsilon + B_s^2}} - sgn(B_s) \right|^2 \leq \left| \frac{|B_s|}{\sqrt{\varepsilon + B_s^2}} + 1 \right|^2 \leq 2^2 = 4 < \infty \text{ for each } \varepsilon > 0 \right).$$

Now

define for each $t \geq 0$ $L_t := \lim_{\varepsilon \to 0+} A_t^\varepsilon = \lim_{\varepsilon \to 0+} \left(g_\varepsilon(B_t) - g_\varepsilon(0) - \int_0^t \frac{B_s}{\sqrt{\varepsilon + B_s^2}} dB_s \right) = |B_t| - \int_0^t \operatorname{sgn}(B_s) dB_s$. And for $0 \leq t_1 < t_2 < +\infty$, $\varepsilon > 0$, we have $A_{t_1}^\varepsilon \leq A_{t_2}^\varepsilon$. By letting $\varepsilon \to 0+$, we obtain that $L_{t_1} \leq L_{t_2}$ a.s.. By Separability of $[0, +\infty)$, we could choose an increasing modification of $(L_t)_{t \geq 0}$. (And we denote by $(L_t)_{t \geq 0}$ again this modification).

3. For $\delta > 0$, $u, v \in (0, +\infty)$, $u < v$, and $\omega \in \{\omega : |B_t| \geq \delta, t \in [u,v]\}$ we have $|A_u^\varepsilon(\omega) - A_v^\varepsilon(\omega)| = \left| \int_u^v \frac{1}{2} \frac{\varepsilon}{\sqrt{\varepsilon + B_s^2(\omega)}} ds \right| \leq \left| \int_u^v \frac{1}{2} \frac{\varepsilon}{\sqrt{\varepsilon + \delta^2}} ds \right| \leq \frac{1}{2}\sqrt{\varepsilon}(v-u)$. By the fact that for each $t \geq 0$, $A_t^\varepsilon \xrightarrow[L^2]{\varepsilon \to 0^+} L_t$, we have that there exist a sequence $(n_k)_{k \in \mathbb{N}^*}$ with $n_k > 0$, $n_k \to 0$ as $k \to +\infty$ such that for a.s. ω, $A_u^{n_k}(\omega) \xrightarrow{k \to +\infty} L_u(\omega)$ and $A_v^{n_k}(\omega) \xrightarrow{k \to +\infty} L_v(\omega)$. By the inequality $|A_u^{n_k}(\omega) - A_v^{n_k}(\omega)| \leq \frac{1}{2}\sqrt{n_k}(v-u)$, we obtain that almost surely $L_u = L_v$. In other words, we obtain that for fixed $\delta > 0$, $0 < u < v < +\infty$,

$$P(\{\omega : |B(\cdot, \omega)| \geq \delta \text{ on } [u,v],\ L_u(\omega) \neq L_v(\omega)\}) = 0.$$

This is the proof of the first part of this question.

For the rest part, firstly note that since $\mathbb{Q} \cap (0, +\infty)$ is countable, the for each $\delta > 0$,

$$P(\{\omega : \exists u < v \in \mathbb{Q} \cap (0, +\infty) \text{ s.t. } |B(\cdot, \omega)| \geq \delta \text{ on } [u,v],\ L_u(\omega) \neq L_v(\omega)\}) = 0,$$

i.e., for each fixed $\delta > 0$, we have that

$$P(\{\omega : \forall u < v \in \mathbb{Q} \cap (0, +\infty), |B(\cdot, \omega)| \geq \delta \text{ on } [u, v], L_u(\omega) = L_v(\omega)\}) = 1.$$

By density of $\mathbb{Q} \cap (0, +\infty)$ in $[0, +\infty)$ and increasing of $(L_t)_{t \geq 0}$, we obtain that for each fixed $\delta > 0$,

$$P(\{\omega : \forall u < v \in [0, +\infty), |B(\cdot, \omega)| \geq \delta \text{ on } [u, v], L_u(\omega) = L_v(\omega)\}) = 1.$$

Thus we obtain that

$$P(\{\omega : \forall u < v \in [0, +\infty), |B(\cdot, \omega)| > 0 \text{ on } [u, v], L_u(\omega) = L_v(\omega)\})$$

$$= P\left(\bigcap_{n=1}^{\infty} \{\omega : \forall u < v \in [0, +\infty), |B(\cdot, \omega)| > \tfrac{1}{n} \text{ on } [u, v], L_u(\omega) = L_v(\omega)\}\right)$$

$= 1$. Since $(L_t)_{t \geq 0}$ is an increasing process, we obtain that $L(\cdot, \omega)$ is a constant in each connected component of $\{t \geq 0, B(t, \omega) \neq 0\}$ for a.s. $\omega \in \Omega$. The proof is done.

4. $\beta_t = \int_0^t \operatorname{sgn}(B_s) dB_s$. Then $(\beta_t)_{t \geq 0}$ is a continuous local martingale. Since for $t \geq 0$, $\langle \beta_t \rangle = \langle \int_0^t \operatorname{sgn}(B_s) dB_s \rangle = \int_0^t 1 \, ds = t$, by Levy's characterization theorem, We know that $(\beta_t)_{t \geq 0}$ is a B.M. It's obvious that it starts from 0.

5. For each fixed $t \geq 0$, and a.s. ω, we have for all $s \in [0,t]$, $L_t \geq L_s = |B_s| - \int_0^s \operatorname{sgn}(B_\tau)\,dB_\tau \geq -\int_0^s \operatorname{sgn}(B_\tau)\,dB_\tau(\omega)$ $= -\beta_s(\omega)$, which means that $L_t \geq \sup_{s \in [0,t]} -\beta_s$ a.s.. On the other hand, for each ω, the set $\{s \in [0,t], B(s,\omega)=0\}$ is a compact subset of $[0,t]$. ~~Let $t_M(\omega) := \max\{s \in$~~ Denote by $s_t(\omega) := \max\{s \in [0,t], B(s,\omega)=0\}$ for each ω. Then we have $|B(\cdot,\omega)| > 0$ on $(s_t(\omega), t]$. By 4) we obtain that for a.s. ω,

$L_t(\omega) = L_{S_t(\omega)+\frac{1}{n}}(\omega)$ for each $n \in \mathbb{N}^*$. Note that $(L_t)_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-submartingale, and $\mathbb{E}[L_t] = \mathbb{E}\left[|B_t| - \int_0^t \mathrm{sgn}(B_s)\,dB_s\right]$ $= \mathbb{E}|B_t| = \sqrt{\frac{2}{\pi}}\sqrt{t}$, thus $t \mapsto \mathbb{E}[L_t]$ is continuous in $t \geq 0$. By Theorem 3.18 of GTM 274, (note that it is assumed above Exercises of this chapter that $(\mathcal{F}_t)_{t\geq 0}$ is complete and the filtration in this Ex 5.32 is larger than the natural one generated by a B.M., which means that $(\mathcal{F}_t)_{t\geq 0}$ is also right-continuous) we obtain that $(L_t)_{t\geq 0}$ has a modification having right-continuous sample paths and meanwhile is still increasing (which could be easily seen from the construction of this modification; see Theorem 3.17 also)

Thus for a.s. ω, $L_t(\omega) = \lim_{n \to +\infty} L_{S_t(\omega)+\frac{1}{n}}(\omega) = L_{S_t(\omega)}(\omega) = |B_{S_t(\omega)}(\omega)| - \beta_{S_t(\omega)}(\omega) = 0 - \beta_{S_t(\omega)}(\omega) \leq \sup_{s \in [0,t]} -\beta_s(\omega)$. Therefore we have $L_t(\omega) = \sup_{s \in [0,t]} -\beta_s(\omega)$ a.s. By 4.) we know that $(-\beta_t)_{t \geq 0}$ is also a B.M. which implies that for $t \geq 0$,

$Law(L_t) = Law(\sup_{s \in [0,t]} -\beta_s) = Law(\sqrt{t}\,|U|)$, where $U \sim N(0,1)$.

6. By results in 3.), we know that $L(\cdot,w)$ is constant if $B(\cdot,w)\neq 0$ in an inteval for a.s. w, which implies that for any integrand $Z(\cdot)$, $a<b\in[0,+\infty)$, we have

$$P\left(\left\{w:\ Z(\cdot,w)=1 \text{ in } (a,b),\ \int_a^b Z_s\,dB_s(w)=B_b(w)-B_a(w)\right\}\right)=1.$$

Then for $t\geqslant 0$, $\varepsilon>0$, $L_t+\int_0^t\sum_{n=1}^{\infty}1_{[S_n^\varepsilon,T_n^\varepsilon]}\,\mathrm{sgn}(B_s)\,dB_s$

$$=|B_t|-\int_0^t \mathrm{sgn}(B_s)\,dB_s+\int_0^t\sum_{n=1}^{\infty}1_{[S_n^\varepsilon,T_n^\varepsilon]}\,\mathrm{sgn}(B_s)\,dB_s$$

$$=|B_t|-\int_0^t \mathrm{sgn}(B_s)\,dB_s+\int_0^t\sum_{n=1}^{N_t^\varepsilon}1_{[S_n^\varepsilon,T_n^\varepsilon]}\,\mathrm{sgn}(B_s)\,dB_s+$$

$$\int_0^t 1_{[S^\varepsilon_{N_t^\varepsilon+1},\ t\vee S^\varepsilon_{N_t^\varepsilon+1}]}\,\mathrm{sgn}(B_s)\,dB_s$$

$$=|B_t|-\int_0^t\left(1_{[0,t]}-\sum_{n=1}^{N_t^\varepsilon}1_{[S_n^\varepsilon,T_n^\varepsilon]}-1_{[S^\varepsilon_{N_t^\varepsilon+1},\ S^\varepsilon_{N_t^\varepsilon+1}\vee t]}\right)\cdot$$

$$\cdot\,\mathrm{sgn}(B_s)\,dB_s$$

$$= |B_t| - \int_0^t \Big(\sum_{n=1}^{N_t^\varepsilon - 1} 1_{(T_n^\varepsilon, S_{n+1}^\varepsilon)}(s) + 1_{(T_{N_t^\varepsilon}^\varepsilon, t\wedge S_{N_t^\varepsilon+1}^\varepsilon)}(s) + 1_{[S_{N_t^\varepsilon+1}^\varepsilon, S_{N_t^\varepsilon+1}^\varepsilon \vee t)}(s)$$

$$- 1_{[S_{N_t^\varepsilon+1}^\varepsilon, S_{N_t^\varepsilon+1}^\varepsilon \wedge t)}(s) \Big) \mathrm{sgn}(B_s)\, dB_s$$

$$= |B_t| - \int_0^t \Big(\sum_{n=1}^{N_t^\varepsilon - 1} 1_{(T_n^\varepsilon, S_{n+1}^\varepsilon)}(s) + 1_{(T_{N_t^\varepsilon}^\varepsilon, t\wedge S_{N_t^\varepsilon+1}^\varepsilon)}(s) \Big) \mathrm{sgn}(B_s)\, dB_s$$

$$= |B_t| - \int_0^t \Big[\sum_{n=1}^{N_t^\varepsilon} 1_{(T_n^\varepsilon, S_{n+1}^\varepsilon)}(s) - 1_{[t\wedge S_{N_t^\varepsilon+1}^\varepsilon, S_{N_t^\varepsilon+1}^\varepsilon)}(s) \Big] \mathrm{sgn}(B_s)\, dB_s$$

$$= |B_t| - \sum_{n=1}^{N_t^\varepsilon} \int_{T_n^\varepsilon}^{S_{n+1}^\varepsilon} \mathrm{sgn}(B_s)\, dB_s + \int_{t\wedge S_{N_t^\varepsilon+1}^\varepsilon}^{S_{N_t^\varepsilon+1}} \mathrm{sgn}(B_s)\, dB_s$$

$$= |B_t| - \sum_{n=1}^{N_t^\varepsilon} (B_{S_{n+1}^\varepsilon} - B_{T_n^\varepsilon}) \mathrm{sgn}(B_{T_n^\varepsilon}) + (B_{S_{N_t^\varepsilon+1}^\varepsilon} - B_t) \mathrm{sgn}(B_t) 1_{\{t < S_{N_t^\varepsilon+1}^\varepsilon\}}$$

$$= |B_t| + \varepsilon N_t^\varepsilon - |B_t| 1_{\{t < S_{N_t^\varepsilon+1}^\varepsilon\}} .$$

$= \varepsilon N_t^\varepsilon + |B_t| 1_{\{t \geq S_{N_t^\varepsilon+1}^\varepsilon\}}$. Set $r_t^\varepsilon := |B_t| 1_{\{t \geq S_{N_t^\varepsilon+1}^\varepsilon\}}$, and note that on $\{(t,\omega): t \geq S_{N_t^\varepsilon+1}^\varepsilon(\omega)\}$, $|B_t|^{(\omega)} \leq \varepsilon$. Thus we have $|r_t^\varepsilon| \leq \varepsilon$. To show $\varepsilon N_t^\varepsilon \xrightarrow[L^2]{\varepsilon \to 0+} L_t$, it suffices to show that $\int_0^t \big(\sum_{n=1}^\infty 1_{[S_n^\varepsilon, T_n^\varepsilon]}(s) \big) \mathrm{sgn}(B_s)\, dB_s \xrightarrow[L^2]{\varepsilon \to 0+} 0$. Firstly note that for $s \geq 0$, $\big(\sum_{n=1}^\infty 1_{[S_n^\varepsilon, T_n^\varepsilon]}(s) \big) \mathrm{sgn}(B_s) \xrightarrow{\varepsilon \to 0+} 0$ a.s. , then $\mathbb{E}\Big[\int_0^t \big(\sum_{n=1}^\infty 1_{[S_n^\varepsilon, T_n^\varepsilon]}(s) \big) \mathrm{sgn}(B_s)\, dB_s \Big]^2$
$= \mathbb{E}\Big[\int_0^t \big(\sum_{n=1}^\infty 1_{[S_n^\varepsilon, T_n^\varepsilon]}(s) \big)^2 ds \Big] \xrightarrow{\varepsilon \to 0+} 0$ by D.C.T. (Since $\big| \sum_{n=1}^\infty 1_{[S_n^\varepsilon, T_n^\varepsilon]}(s) \big| \leq 1$ for $\varepsilon > 0$).
The proof is done .

7. By 5.) and 6.) , it suffices to show that $\frac{1}{\sqrt{t}} \int_0^t \big(\sum_{n=1}^\infty 1_{[S_n^1, T_n^1]}(s) \big) \mathrm{sgn}(B_s)\, dB_s \xrightarrow[t \to +\infty]{L^2} 0$.

(But I haven't come up with a solution to it yet . Apologize for that . I will upload a solution to this question as soon as I figure it out .)

Ex 5.33

1. By Itô's formula, $|B_t|^2 = |x|^2 + \sum_{i=1}^{N} \int_0^t 2 B_s^i \, dB_s^i + \frac{1}{2} \sum_{i=1}^{N} \int_0^t 2\, dt = |x|^2 + \sum_{i=1}^{N} \int_0^t 2 B_s^i \, dB_s^i + Nt$. And since $\left(\sum_{i=1}^{N} \int_0^t 2 B_s^i \, dB_s^i \right)_{t \geq 0}$ is a continuous local martingale, $(Nt)_{t \geq 0}$ is a continuous finite variation process, thus $(|B_t|^2)_{t \geq 0}$ is a semi-martingale.

And by the fact that for $t \geq 0$, $\mathbb{E} \langle \sum_{i=1}^{N} \int_0^t 2 B_s^i \, dB_s^i \rangle$

$$= \mathbb{E}\left[\sum_{i=1}^{N} \langle \int_0^t 2 B_s^i \, dB_s^i \rangle \right] = \sum_{i=1}^{N} \mathbb{E} \int_0^t 4 |B_s^i|^2 \, ds = \sum_{i=1}^{N} 4 \int_0^t \mathbb{E} |B_s^i|^2 \, ds$$

$$= 4 \sum_{i=1}^{N} \int_0^t s \, ds = 2Nt^2 < \infty,$$

we obtain that $\left(\sum_{i=1}^{N} \int_0^t 2 B_s^i \, dB_s^i \right)_{t \geq 0}$ is a true martingale.

2. For each $i \in \{1, \cdots, N\}$, $t \geq 0$, $\int_0^t \left(\frac{B_s^i}{|B_s|} \right)^2 ds = \int_0^t \frac{{B_s^i}^2}{|B_s|^2} \, ds \leq \int_0^t 1 \, ds = t < \infty$. Thus the stochastic integral in the definition of β_t is well-defined. And $(\beta_t)_{t \geq 0}$ is a continuous local martingale. Since $\langle \beta_t \rangle = \langle \sum_{i=1}^{N} \int_0^t \frac{B_s^i}{|B_s|} \, dB_s^i \rangle = \sum_{i=1}^{N} \langle \int_0^t \frac{B_s^i}{|B_s|} \, dB_s^i \rangle$

$$= \sum_{i=1}^{N} \int_0^t \frac{{B_s^i}^2}{|B_s|^2} \, ds = \int_0^t \sum_{i=1}^{N} \frac{{B_s^i}^2}{|B_s|^2} \, ds = t$$

for $t \geq 0$, thus $(\beta_t)_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$-B.M. starting from 0.

3. (By 1) It suffices to show that $\sum_{i=1}^{N}\int_0^t B_s^i\,dB_s^i = \int_0^t |B_s|\,d\beta_s$ for $t\geq 0$. Actually $\int_0^t |B_s|\,d\beta_s = \int_0^t |B_s|\cdot\sum_{i=1}^{N}\frac{B_s^i}{|B_s|}\,dB_s^i = \sum_{i=1}^{N}\int_0^t B_s^i\,dB_s^i$.

4. Define $S_R := \inf\{t\geq 0 : |B_t|\geq R\}$ for $R>|x|$ (as in next Question). Then for each $n\in\mathbb{N}^*$, $f(|B_{t\wedge T_\varepsilon\wedge S_{n+|x|}}|)$ is integrable and $\mathcal{F}_t$-measurable for both $N=2$ and $N\geq 3$. To prove that $(f(|B_{t\wedge T_\varepsilon\wedge S_{n+|x|}}|))_{t\geq 0}$ is a martingale for each $n\in\mathbb{N}^*$, it suffices to show that for $0\leq s<t<+\infty$,

$$\mathbb{E}\left[f(|B_{t\wedge T_\varepsilon\wedge S_{n+|x|}}|)\,\middle|\,\mathcal{F}_s\right] = f(|B_{s\wedge T_\varepsilon\wedge S_{n+|x|}}|),$$

which is a direct consequence of fundamental Theorem of Calculus and the fact that $f(|x|)$ is harmonic in $x\in\mathbb{R}^N\setminus B(0,\varepsilon)$.

For a detailed proof of a generalized result, see the following Theorem:

(17.1) THEOREM. *Suppose that* $f:\mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is $C^{1,2}$, *and that there exists a constant* K *such that, for all* $t \geqslant 0$, $x \in \mathbb{R}^d$,

$$(17.2)\qquad |f(t,x)| + \left|\frac{\partial f}{\partial t}(t,x)\right| + \sum_{j=1}^{d}\left|\frac{\partial f}{\partial x_j}(t,x)\right| + \sum_{i=1}^{d}\sum_{j=1}^{d}\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x)\right| \leqslant K e^{K(t+|x|)}$$

Then the process

$$(17.3)\qquad C_t^f := f(t,B_t) - f(0,B_0) - \int_0^t \mathscr{G} f(s,B_s)\,ds \quad \text{is a martingale,}$$

where

$$(17.4)\qquad \mathscr{G} f(t,x) := \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{j=1}^{d}\frac{\partial^2 f}{\partial x_j^2}\right)(t,x).$$

Remarks. The class $C^{1,2}$ is, of course, the class of functions $f(t,x)$ with continuous partial derivatives of all orders up to 1 in t and up to 2 in x. The exponential growth condition (17.2) will be seen to be unnecessary provided we relax the statement (17.3) to say that C^f is a *local* martingale. We shall not digress to define this now. In dimension $d=1$, the only functions of x for which $f(B_t)$ is a martingale are the linear functions, but in dimension $d \geqslant 2$ we shall see that there is a very rich family of f for which $f(B_t)$ is a martingale.

Proof. We must prove that, for $0 \leqslant s \leqslant t$,

$$\mathbf{E}[C_t^f - C_s^f \mid \mathscr{F}_s] = 0,$$

for which, by the independent-increments property of B, it will suffice to prove that, for any $x \in \mathbf{R}^d$ and $t \geqslant 0$

$$\mathbf{E}^x[C_t^f] = 0, \tag{17.5}$$

where $\mathbf{P}^x$ is the law of Brownian motion started at x. Without loss of generality, we can take $x = 0$ (and write $\mathbf{P}$ for $\mathbf{P}^0$), and we shall prove that, for $0 < \varepsilon < t$,

$$\mathbf{E}[C_t^f - C_\varepsilon^f] = 0. \tag{17.6}$$

Using the assumption (17.2), the fact that $\mathbf{P}[\sup_{u \leqslant t} |B_u| \geqslant a] \leqslant c\mathbf{P}[|B_1| \geqslant a/\sqrt{t}]$ (see (13.4)), and dominated convergence, (17.6) implies (17.5).

Letting $p_t(x) := (2\pi t)^{-d/2} \exp(-|x|^2/2t)$ denote the d-dimensional Brownian transition density, we observe that, for $t > 0$, $x \in \mathbf{R}^d$,

$$\frac{\partial p_t}{\partial t}(x) = \frac{1}{2} \sum_{j=1}^{d} \frac{\partial^2 p_t}{\partial x_j^2}(x). \tag{17.7}$$

Hence

$$\begin{aligned}
\mathbf{E}[C_t^f - C_\varepsilon^f] &= E\left[f(t, B_t) - f(\varepsilon, B_\varepsilon) - \int_\varepsilon^t \mathscr{G} f(s, B_s)\, ds \right] \\
&= \int [p_t(x) f(t, x) - p_\varepsilon(x) f(\varepsilon, x)]\, dx \\
&\quad - \int_\varepsilon^t ds \int p_s(x) \left[\frac{\partial f}{\partial t}(s, x) + \tfrac{1}{2} \Delta f(s, x) \right] dx.
\end{aligned}$$

But

$$\int p_s(x) \tfrac{1}{2} \Delta f(s, x)\, dx = \int \tfrac{1}{2} \Delta p_s(x) f(s, x)\, dx,$$

(integrating twice by parts and using (17.2))

$$= \int \frac{\partial p_s}{\partial t}(x) f(s, x)\, dx,$$

using (17.7). Thus

$$\begin{aligned}
\mathbf{E}[C_t^f - C_\varepsilon^f] &= \int [p_t(x) f(t, x) - p_\varepsilon(x) f(\varepsilon, x)]\, dx \\
&\quad - \int_\varepsilon^t ds \int \left[p_s(x) \frac{\partial f}{\partial t}(s, x) + f(s, x) \frac{\partial p_s}{\partial t}(x) \right] dx \\
&= \int [p_t(x) f(t, x) - p_\varepsilon(x) f(\varepsilon, x)]\, dx - \int_\varepsilon^t ds \int \frac{\partial}{\partial t}(p_s(x) f(s, x))\, dx \\
&= \int [p_t(x) f(t, x) - p_\varepsilon(x) f(\varepsilon, x)]\, dx - \int \left\{ \int_\varepsilon^t ds \frac{\partial}{\partial t} [p_s(x) f(s, x)] \right\} dx \\
&= 0.
\end{aligned}$$

□

5. Since $|f(|B_{t\wedge T_\varepsilon\wedge S_R}|)|$ is bounded ~~for~~ in $t\geq 0$ provided that $|x|>\varepsilon$ and $|R|>|x|$, then $(f(|B_{t\wedge T_\varepsilon\wedge S_R}|))_{t\geq 0}$ is UI. By O.S.T., we have that $\mathbb{E}[f(|B_{0\wedge T_\varepsilon\wedge S_R}|)]$

$= \mathbb{E}[f(|B_{T_\varepsilon\wedge S_R\wedge T_\varepsilon\wedge S_R}|)]$, which implies that

$$f(|x|) = \mathbb{E}[f(R)1_{\{T_\varepsilon\geq S_R\}} + f(\varepsilon)1_{\{T_\varepsilon<S_R\}}]$$

$$= f(R)\cdot(1-P(T_\varepsilon<S_R)) + f(\varepsilon)P(T_\varepsilon<S_R).$$

Therefore $P(T_\varepsilon<S_R) = \dfrac{f(R)-f(|x|)}{f(R)-f(\varepsilon)}$.

Note that $(T_\varepsilon)_{\varepsilon>0}$ is increasing ~~in $\varepsilon\to 0$~~ as $\varepsilon\to 0$. Thus we have

$$P(T_0<S_R) = P\Big(\bigcap_{n=\frac{1}{|x|}}^{\infty}\{T_{\frac{1}{n}}<S_R\}\Big) = \lim_{n\to+\infty}\frac{f(R)-f(|x|)}{f(R)-f(\frac{1}{n})}$$

$=0$. Therefore for each $n\in\mathbb{N}^*$, $\exists\, \Omega_n\subseteq\Omega$ s.t. $P(\Omega_n)=1$ and $T_0(\omega)\geq S_n(\omega)$ for $\omega\in\Omega_n$. Set $\widetilde{\Omega} := \bigcap_{n=1}^{\infty}\Omega_n$. Then $P(\widetilde{\Omega})=1$, and for $\omega\in\widetilde{\Omega}$, and for any $t\geq 0$, $\sup_{s\in[0,t]}|B_s|(\omega) < \infty$ (since $|B(\cdot,\omega)|$ is continuous), ~~set~~ let $n>\sup_{s\in[0,t]}|B_s|(\omega)$. Then we have $T_0(\omega)\geq S_{n(\omega)}(\omega)$, which implies that $|B_t(\omega)|\leq\sup_{[0,t]}|B_s(\omega)| < n$, and ~~thus~~ $t<S_{n(\omega)}(\omega)$, thus $B_t(\omega)\neq 0$. The proof is done.

6. By Itô's formula, $|B_t| = |B_0| + \sum_{i=1}^{N} \int_0^t \frac{B_s^i}{|B_s|} dB_s^i + \frac{1}{2} \sum_{i=1}^{N} \int_0^t \frac{|B_s|^2 - {B_s^i}^2}{|B_s|^3} d\langle B_s^i \rangle = |x| + \beta_t + \frac{1}{2} \int_0^t \frac{N}{|B_s|} - \frac{|B_s|^2}{|B_s|^3} ds$
$= |x| + \beta_t + \frac{1}{2}(N-1) \int_0^t \frac{ds}{|B_s|}$.

7. By 5 we know that a.s. $T_0(\omega) = +\infty$. Define $\tilde{T}_n := T_{\frac{1}{n}} \nearrow +\infty$ a.s. as $n \to +\infty$. Since for $t \geq 0$ $|B_{t\wedge \tilde{T}_n}|^{2-N}$ is bounded for each $n \in \mathbb{N}^*$, ~~thus~~ and we could show as in 4 that $(|B_{t\wedge\tilde{T}_n}|^{2-N})_{t\geq 0}$ is a martingale. Thus $(|B_t|^{2-N})_{t \geq 0}$ is a local martingale. Since it is non-negative then it is an nonnegative supermartingale (by Fatou's Lemma) (And the fact that $|B_0|^{2-N} = |x|^{2-N}$ is integrable). By martingale Convergence theorem, there exists a r.v. $Z \in \mathcal{F}_\infty$ s.t. $\lim_{t\to+\infty} |B_t|^{2-N} = Z$ a.s. And by 5 we know that for a.s. ω, and any $R > |x|$, ~~$P(S_R < +\infty)$~~ $S_R(\omega) < +\infty$, which implies that $Z \equiv 0$ a.s. Therefore we obtain that $\lim_{t\to+\infty} |B_t| = +\infty$ a.s.

8. Set $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. For each $t \geq 0$ $\mathbb{E}|B_t|^{-2} = \mathbb{E}\Big[|B_t|^{-2} \mathbb{1}_{\{|B_t| > \frac{1}{2}|x|\}} + |B_t|^{-2} \mathbb{1}_{\{|B_t| \leq \frac{1}{2}|x|\}}\Big] \leq 4|x|^{-2} + \mathbb{E}\Big[|B_t|^{-2} \mathbb{1}_{\{|B_t| \leq \frac{1}{2}|x|\}}\Big]$.

And $\mathbb{E}\left[|B_t|^{-2} 1_{\{|B_t| \le \frac{1}{2}|x|\}}\right] = \int_{\mathbb{R}^3} \frac{1}{|y|^2} 1_{\{|y| \le \frac{1}{2}|x|\}} (2\pi t)^{-\frac{3}{2}} e^{-\frac{|y-x|^2}{2t}} dy$

$$= (2\pi t)^{-\frac{3}{2}} \int_{B(0,\frac{1}{2}|x|)} |y|^{-2} e^{-\frac{|y-x|^2}{2t}} dy \le (2\pi t)^{-\frac{3}{2}} \int_{B(0,\frac{1}{2}|x|)} |y|^{-2} e^{-\frac{(|x|-\frac{1}{2}|x|)^2}{2t}} dy$$

$$= (2\pi t)^{-\frac{3}{2}} \cdot e^{-\frac{|x|^2}{8t}} \cdot \int_{B(0,\frac{1}{2}|x|)} |y|^{-2} dy$$

$$= (2\pi t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{8t}} \cdot \int_0^{\frac{1}{2}|x|} \int_0^{2\pi} \int_0^{\pi} (r^2\cos^2\varphi \sin^2\theta + r^2 \sin^2\varphi \sin^2\theta + r^2\cos^2\theta)^{-1} r^2 \sin\theta \, d\theta \, d\varphi \, dr$$

$$= (2\pi t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{8t}} \int_0^{\frac{1}{2}|x|} \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\varphi \, dr = 2\pi |x| (2\pi t)^{-\frac{3}{2}} \cdot e^{-\frac{|x|^2}{8t}}.$$

$$= C_1 t^{-\frac{3}{2}} e^{-C_2 \frac{1}{t}} \quad (C_1, C_2 \in \mathbb{R}_+) . \quad (*)$$

On $\{t : t \geqslant 1\}$, $(*) \leqslant C_1$; And note that

$$\lim_{t\to 0+} t^{-\frac{3}{2}} e^{-C_2 t^{-1}} = \lim_{t\to+\infty} \frac{t^{\frac{3}{2}}}{e^{C_2 t}} = \lim_{t\to+\infty} \frac{\frac{3}{4} t^{-\frac{1}{2}}}{C_2^2 e^{C_2 t}} = 0,$$

thus $\exists\ \delta \in (0,1)$ s.t. $|(*)| \leqslant 1$ on $\{t : t \in [0, \delta]\}$.

By continuity of $(*)$ in $t \geqslant 0$, we know that $(*)$ is bounded on $\{t : t \in [\delta, 1]\}$. Therefore we conclude that $(*)$ is bounded on $\{t : t \in [0, +\infty)\}$.

Then we obtain that $(|B_t|^{-1})_{t\geq 0}$ is L^2 bounded in t, which implies that $(|B_t|^{-1})_{t\geq 0}$ is UI continuous local martingale. Together with results of 7, we know that $|B_t|^{-1} \xrightarrow[L^1]{t\to+\infty} 0$, which means $\mathbb{E}\left||B_t|^{-1}-0\right| = \mathbb{E}|B_t|^{-1}$ $\longrightarrow 0$ as $t\to+\infty$. Thus $(|B_t|^{-1})_{t\geq 0}$ is not a true martingale, otherwise we would have that

$$\mathbb{E}|B_t|^{-1} \equiv |x|^{-1} \text{ for all } t\geq 0.$$