HW 5

EX 1

(1). $Z_t = e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$, $t \geq 0$, $\lambda \in \mathbb{R}$. $T_a = \inf\{t \geq 0, B_t = a\}$.

$a > 0$. Since $Z_{t\wedge T_a} = e^{\lambda B_{t\wedge T_a} - \frac{1}{2}\lambda^2 t\wedge T_a}$

$\leq e^{\lambda B_{t\wedge T_a}} \leq e^{\lambda a}$ for $\lambda \geq 0$, and $\mathbb{E} e^{\lambda a} = e^{\lambda a} < \infty$,

then we obtain that for $\lambda \geq 0$, $(Z_{t\wedge T_a})_{t\geq 0}$ is UI.

By O.S.T. we have that

$$\mathbb{E}\left[e^{\lambda a} \cdot e^{-\frac{1}{2}\lambda^2 T_a}\right] = \mathbb{E}\left[Z_{T_a \wedge T_a}\right] = \mathbb{E}\left[Z_{0\wedge T_a}\right] = 1,$$

which implies that $\mathbb{E}\left[e^{-\frac{1}{2}\lambda^2 T_a}\right] = e^{-\lambda a}$ for $\lambda \geq 0$.

Take $\lambda = \sqrt{2c} \geq 0$ for $c > 0$, we obtain that

$$\mathbb{E} e^{-cT_a} = e^{-\sqrt{2c}\,a}.$$

(2). Since $Z_t \geq 0$ for $t \geq 0$ and $(Z_t)_{t\geq 0}$ is a supermartingale, we know that $\exists$ r.v. Z_∞ s.t. $Z_t \to Z_\infty$ a.s. as $t \to +\infty$.

And for $\varepsilon, M > 0$, and $n \in \mathbb{N}^*$, we have that

$$P\Big(\sup_{t\in[0,n]} B_t > \varepsilon n + M\Big) = P(|B_n| > \varepsilon n + M)$$

$$= P\left(\left|\frac{B_n}{\sqrt{n}}\right| > \varepsilon\sqrt{n} + \frac{M}{\sqrt{n}}\right) \leq e^{-\frac{1}{2}\left(\varepsilon\sqrt{n} + \frac{M}{\sqrt{n}}\right)^2}$$

$\le e^{-\frac{1}{2}\varepsilon^2 n}$. Since $\sum_{n=1}^{\infty} e^{-\frac{1}{2}\varepsilon^2 n} < \infty$, we obtain by Borel-Cantelli Lemma that

$$P\left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\left\{\sup_{t\in[0,n]} B_t \le \varepsilon n + M\right\}\right) = 1,$$

which implies that for a.s. $\omega\in\Omega$, $\exists\, n_0(\omega)\in\mathbb{N}^*$ s.t. for $n > n_0(\omega)$, $\sup_{t\in[0,n]} B_t \le \varepsilon n + M$.

Therefore with probability 1, $\exists\, t_0(\omega) > 0$ s.t. for $t > t_0(\omega)$,
$\cancel{B_t(\omega) \le \sup_{t\in[0,n]} B_t}$ $B_t(\omega) \le \varepsilon([t]+1) + M$.

Thus for sufficiently large $t > 0$, $Z_t = e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$
$\le e^{\lambda(\varepsilon[t]+\varepsilon+M) - \frac{1}{2}\lambda^2 t} \le e^{\lambda((\varepsilon-\frac{1}{2}\lambda)[t]) + \lambda(\varepsilon+M)}$.

If we take $\varepsilon = \frac{1}{4}\lambda$ for example, then we have that $Z_t \to 0$ a.s. as $n\to+\infty$.

OR: You could prove this result via Law of Iterative Logarithm: $\limsup_{t\to+\infty} \frac{B_t}{(2t(\log\log t))^{\frac{1}{2}}} = 1$.

(3). $S_a = \inf\{t \ge 0, B_t \ge at+1\}$ is a stopping time for $a > 0$.

~~(since $\{B_t \ge at+1\}$ is progressive}). (Maybe some completion is needed for the filtration)~~

For each $t \ge 0$, $S_a \wedge t$ is a bounded stopping time.

Then by O.S.T., $\mathbb{E}[Z_{S_a\wedge t}] = \mathbb{E}[Z_0] = 1$.

But note that for $\omega \in \{S_a \leq t\}$, $S_a\wedge t(\omega) = S_a$, and $B_{S_a} = aS_a + 1$; And for $\omega\in\{S_a > t\}$, $S_a \wedge t(\omega) = t$.

Set $\lambda = 2a$, we have that $\mathbb{E}[Z_{S_a\wedge t}] = \mathbb{E}\Big[Z_{S_a}1_{\{S_a\leq t\}} + Z_t 1_{\{S_a>t\}}\Big] = \mathbb{E}\Big[e^{2a(aS_a+1)-2a^2S_a}1_{\{S_a\leq t\}} + e^{2aB_t-2a^2t}1_{\{S_a>t\}}\Big] = \mathbb{E}\Big[e^{2a}1_{\{S_a\leq t\}} + e^{2aB_t-2a^2t}1_{\{S_a>t\}}\Big]$.

Therefore we obtain that

$$\mathbb{E}\Big[e^{2a}\cdot 1_{\{S_a\leq t\}} + e^{2aB_t-2a^2t}1_{\{S_a>t\}}\Big] = 1,$$

which implies that:

$$\cancel{P(S_a\leq t) = e^{-2a}\Big[1-\mathbb{E}\big[e^{B_t-at}\cdot 1_{\{S_a>t\}}\big]\Big].}$$

~~Then~~ $\cancel{\big|P(S_a\leq t) - e^{-2a}\big| = \mathbb{E}\big[e^{B_t-at}\cdot 1_{\{S_a>t\}}\big]}$

$$\cancel{\leq \Big(\mathbb{E}\, e^{2B_t-2at}\cdot \mathbb{E}\, 1_{\{S_a>t\}}\Big)^{\frac{1}{2}}}$$

$P(S_a < +\infty) = e^{-2a}$ by letting $t\to+\infty$.

(Since $1_{\{S_a\leq t\}} \xrightarrow{t\to+\infty} 1_{\{S_a<+\infty\}}$ in L^1, and also $e^{2aB_t-2a^2t} \to 0$ as $t\to+\infty$ a.s., and $e^{2aB_t-2a^2t}1_{\{S_a>t\}} \leq e^{2a(at+1)-2a^2t}1_{\{S_a>t\}} = e^{2a}1_{\{S_a>t\}}$. By D.C.T.,

$e^{2aB_t - 2a^2 t} \mathbb{1}_{\{S_a > t\}} \xrightarrow{n \to +\infty} 0$ in L^1.)

Ex 2

(1). For each $t \in \mathbb{Q} \cap [0, +\infty)$, let Y_t be a version of $\mathbb{E}[X_t | \mathcal{G}]$. By continuity of $(X_t)_{t \geq 0}$, we know that for a.s. $\omega \in \Omega$, $X(\cdot, \omega)$ is continuous at each $t \geq 0$. Then for $t \geq 0$, and any sequence $(t_n)_{n \geq 0}$ with $t_n \in \mathbb{Q} \cap [0, +\infty)$, $t_n \xrightarrow{n \to +\infty} t$, we have $X(t, \omega) = \lim_{n \to +\infty} X(t_n, \omega)$. Then by Conditional D.C.T. and boundedness of $(X_t)_{t \geq 0}$, we have that $\mathbb{E}[X_t | \mathcal{G}] = \lim_{n \to +\infty} \mathbb{E}[X_{t_n} | \mathcal{G}]$ a.s.

Now let me show that for a.s. $\omega \in \Omega$, the sample path $Y(\cdot, \omega)$ on $\mathbb{Q} \cap [0, +\infty)$ is continuous with a modulus of continuity not larger than that of $X(\cdot, \omega)$. Define for $\delta > 0$
$\omega(\delta, X^{(\omega)}) := \sup\{|X_t - X_s|(\omega), |t-s| \leq \delta, t, s \in \mathbb{Q} \cap [0, +\infty)\}$, and $\omega(\delta, Y(\omega))$ similarly. By the fact that for $s, t \in \mathbb{Q} \cap [0, +\infty)$, $\cancel{|Y(t) - Y(s)| = \mathbb{E}[X_t - X_s}$

$$|Y_t - Y_s| = |\mathbb{E}[X_t | \mathcal{G}] - \mathbb{E}[X_s | \mathcal{G}]| = |\mathbb{E}[X_t - X_s | \mathcal{G}]|$$

$\leq \mathbb{E}[|X_t - X_s| | \mathcal{G}]$ for a.s. ω (since $\mathbb{Q} \cap [0, +\infty)$ is countable)

then we have for $|t-s| \leq \delta$, $|Y_t - Y_s| \leq \mathbb{E}[\omega(\delta, X(\cdot)) | \mathcal{G}](\omega)$ ($\omega(\delta, X(\cdot)) \in L^1$ by boundedness of $X(\cdot)$).

Then by Conditional Fatou's Lemma, we obtain that

$$(*)\quad \limsup_{\delta\to 0+}\ \sup_{|t-s|\le\delta} |Y_t - Y_s| \le \mathbb{E}\Big[\limsup_{\delta\to 0+} \omega(\delta, X(\cdot)) \,\Big|\, \mathcal{G}\Big] \text{ a.s.}.$$

Now let me replace $\mathbb{Q}\cap[0,+\infty)$ with $\mathbb{Q}\cap[n-1,n]$ for each $n\in\mathbb{N}^*$ in $(*)$, by compactness of $[n-1,n]$ and continuity of $X(\cdot)$, we obtain that $X(\cdot)$ is a.s. uniformly continuous on $[n-1,n]$ and therefore for each $n\in\mathbb{N}^*$, $\exists\ \Omega_n\subseteq\Omega$ s.t. $P(\Omega_n)=1$ with $Y(\omega)$ is uniformly continuous on $\mathbb{Q}\cap[n-1,n]$. By density of $\mathbb{Q}\cap[n-1,n]$ in $[n-1,n]$, there exists an unique continuous extension $Y^{(n)}(t,\omega)$ with $t\in[n-1,n]$ for all $\omega\in\Omega_n$. Therefore, for $\omega\in\bigcap_{n=1}^{\infty}\Omega_n$, which has probability one ~~that~~ we have that $Y(t,\omega) := \sum_{n=1}^{\infty} Y^{(n)}(t,\omega)\mathbb{1}_{\{[n-1,n)\}}$ is continuous at $t\geq 0$. Let $Y(t,\omega)=0$ for $\omega\notin\bigcap_{n=1}^{\infty}\Omega_n$, $t\geq 0$, $t\notin\mathbb{Q}$.

~~The boundedness of $Y(\cdot)$ is obtained by boundedness of $(Y_t)_{t\in\mathbb{Q}\cap[0,+\infty)}$~~

For each $t\geq 0$, $\exists\ (t_n)_{n\in\mathbb{N}^*}\nearrow t$, $t_n<t$, $t_n\in\mathbb{Q}$, then

$$Y_t = \lim_{n\to+\infty} Y_{t_n} = \lim_{n\to+\infty}\mathbb{E}[X_{t_n}\,|\,\mathcal{G}] = \mathbb{E}[X_t\,|\,\mathcal{G}] \text{ a.s.}.$$

The boundedness of $Y(\cdot)$ is then obtained via boundedness of $X(\cdot)$.

The proof of (1) is done.

2. For each partition Δ, $\mathbb{E}\left[\int_0^t X_s^\Delta ds \mid \mathcal{G}\right]$

$$= \mathbb{E}\left[\int_0^t \sum_{k=0}^{n-1} X_{t_k} \mathbb{1}_{[t_k, t_{k+1})}(s) ds \mid \mathcal{G}\right] = \sum_{k=0}^{n-1} (t_{k+1} - t_k) \mathbb{E}[X_{t_k} \mid \mathcal{G}]$$

$$= \int_0^t \sum_{k=0}^{n-1} \mathbb{E}[X_{t_k} \mid \mathcal{G}] \mathbb{1}_{[t_k, t_{k+1})}(s) ds$$

$= \int_0^t \mathbb{E}[X(\cdot) \mid \mathcal{G}]_s^\Delta ds$. By **1**, we know that there exists a modification of $\mathbb{E}[X(\cdot) \mid \mathcal{G}]$ that is continuous, which means that there exists a bounded continuous process $Y(\cdot)$ with $Y_t = \mathbb{E}[X_t \mid \mathcal{G}]$ and $\lim_{|\Delta| \to 0} \int_0^t Y_s^\Delta ds = \int_0^t Y_s ds$ for a.s. $\omega \in \Omega$. Thus we have that $\mathbb{E}\left[\int_0^t X_s ds \mid \mathcal{G}\right]$

$$= \mathbb{E}\left[\lim_{|\Delta| \to 0} \int_0^t X_s^\Delta ds \mid \mathcal{G}\right] = \lim_{|\Delta| \to 0} \mathbb{E}\left[\int_0^t X_s^\Delta ds \mid \mathcal{G}\right]$$

$$= \lim_{|\Delta| \to 0} \int_0^t \mathbb{E}[X(\cdot) \mid \mathcal{G}]_s^\Delta ds = \lim_{|\Delta| \to 0} \int_0^t Y_s^\Delta ds$$

$= \int_0^t Y_s ds = \int_0^t \mathbb{E}[X_s \mid \mathcal{G}] ds$, by D.C.T. and boundedness of $(X_t)_{t \geq 0}$.

3. For $0 \le s < t < +\infty$, we have that

$$\mathbb{E}\left[e^{i\lambda B_t} + \int_0^t \frac{1}{2}\lambda^2 e^{i\lambda B_\tau} d\tau \,\middle|\, \mathcal{F}_s\right]$$

$$= \mathbb{E}\left[e^{i\lambda B_s} \cdot e^{i\lambda(B_t - B_s)} \,\middle|\, \mathcal{F}_s\right] + \mathbb{E}\left[\int_0^s \frac{1}{2}\lambda^2 e^{i\lambda B_\tau} d\tau + \int_s^t \frac{1}{2}\lambda^2 \right.$$

$$\left. e^{i\lambda B_s} \cdot e^{i\lambda(B_\tau - B_s)} d\tau \,\middle|\, \mathcal{F}_s\right]$$

$$= e^{i\lambda B_s} \cdot \mathbb{E}\, e^{i\lambda(B_t - B_s)} + \int_0^s \frac{1}{2}\lambda^2 e^{i\lambda B_\tau} d\tau + \frac{1}{2}\lambda^2 e^{i\lambda B_s} \cdot$$

$$\mathbb{E}\left[\int_s^t e^{i\lambda(B_\tau - B_s)} d\tau \,\middle|\, \mathcal{F}_s\right]$$

$$= e^{i\lambda B_s} \cdot e^{-\frac{1}{2}\lambda^2 (t-s)} + \int_0^s \frac{1}{2}\lambda^2 e^{i\lambda B_\tau} d\tau + \frac{1}{2}\lambda^2 e^{i\lambda B_s} \cdot$$

$$\int_s^t \mathbb{E}\left[e^{i\lambda(B_\tau - B_s)}\right] d\tau$$

$$= e^{i\lambda B_s} \cdot e^{-\frac{1}{2}\lambda^2 (t-s)} + \int_0^s \frac{1}{2}\lambda^2 e^{i\lambda B_\tau} d\tau + \frac{1}{2}\lambda^2 e^{i\lambda B_s} \cdot$$

$$\int_s^t e^{-\frac{1}{2}\lambda^2 (\tau - s)} d\tau = e^{i\lambda B_s} \cdot e^{-\frac{1}{2}\lambda^2 (t-s)} + \int_0^s \frac{1}{2}\lambda^2 e^{i\lambda B_\tau} d\tau$$

$$+ \frac{1}{2}\lambda^2 e^{i\lambda B_s} \cdot e^{\frac{1}{2}\lambda^2 s} \cdot \frac{1}{-\frac{1}{2}\lambda^2}\left(e^{-\frac{1}{2}\lambda^2 t} - e^{-\frac{1}{2}\lambda^2 s}\right)$$

$$= e^{i\lambda B_s} + \int_0^s \frac{1}{2}\lambda^2 e^{i\lambda B_\tau} d\tau.$$

Integrability is obvious since $|e^{i\theta}| = 1$ for $\theta \in \mathbb{R}$.

Adaptedness is obvious by definition of the integral.

Thus the proof is done.

Ex 3

For each $n \in \mathbb{N}^*$, $\Delta_n : it \cdot 2^{-n}$, $0 \leq i \leq 2^n$. Define for $t > 0$ that $V_t^{(n)} := \sum_{i=0}^{2^n-1} (B_{t_{i+1}} - B_{t_i})^2$. Since $\{(B_{t_{i+1}} - B_{t_i})\}_{i=0}^{2^n-1}$ are i.i.d and have the distribution $N(0, t_{i+1} - t_i) = N(0, \frac{t}{2^n})$, then we have that

$$\mathbb{E} V_t^{(n)} = 2^n \cdot \mathbb{E}(B_{t_1} - B_{t_0})^2 = 2^n \cdot \frac{t}{2^n} = t.$$

And

$$\operatorname{Var} V_t^{(n)} = \operatorname{Var}\Big(\sum_{i=0}^{2^n-1} (B_{t_{i+1}} - B_{t_i})^2\Big) = 2^n \cdot \operatorname{Var}\big[(B_{t_1} - B_{t_0})^2\big]$$
$$= 2^n \cdot 2\left(\frac{t}{2^n}\right)^2 = \frac{t^2}{2^{n-1}}$$

By Markov's inequality, we have for $n \in \mathbb{N}^*$, $\varepsilon > 0$:

$$P(|V_t^{(n)} - t| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} \cdot \mathbb{E}|V_t^{(n)} - t|^2$$
$$= \frac{1}{\varepsilon^2} \cdot \operatorname{Var} V_t^{(n)} = \frac{1}{\varepsilon^2} \cdot \frac{t^2}{2^{n-1}}.$$

Then we have

$$\sum_{n=1}^{\infty} P(|V_t^{(n)} - t| \geq \varepsilon) \leq \sum_{n=1}^{\infty} \frac{1}{\varepsilon^2} \cdot \frac{t^2}{2^{n-1}} < \infty.$$

By Borel-Cantelli Lemma, $P\Big(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{|V_t^{(n)} - t| < \varepsilon\}\Big) = 1$, which implies the desired result.