HW5

March 20, 2024

Exercise 1 Recall that $(Z_t = e^{\lambda B_t - \frac{\lambda^2}{2} t})_{t \geq 0}$ is a martingale for every $\lambda \in \mathbb{R}$. It is a non-negative super-martingale so it has an almost sure limit Z_∞.

1. Let T_a be the hitting time of $a > 0$. Use the Optional Sampling Theorem to show that
$$\mathsf{E} e^{-cT_a} = e^{-a\sqrt{2c}}, \quad c > 0.$$

2. Show that $Z_\infty = 0$.
Hint: One possible proof is to use Borel–Cantelli to show that for any $\varepsilon, M > 0$, $B_n \leq \varepsilon n + M$ for large enough n.

3. Let $S_a = \inf\{t \geq 0 : B_t \geq at + 1\}$, $a > 0$. Use the Optional Sampling Theorem to show that
$$\mathsf{E} e^{2a} \mathbb{1}_{\{S_a \leq t\}} + \mathsf{E} e^{2aB_t - 2a^2 t} \mathbb{1}_{\{S_a > t\}} = 1.$$
Take the limit $t \to \infty$ and show that $\mathsf{P}(S_a < \infty) = e^{-2a}$.

Exercise 2 Let $(X_t)_{t \geq 0}$ be a $(\mathcal{F}_t)$-adapted, bounded continuous process. For any partition $\Delta : 0 = t_0 < t_1 < \cdots < t_n = t$ we define $X_s^\Delta = \sum_{k=0}^{n-1} X_{t_k} \mathbb{1}_{[t_k, t_{k+1})}(s)$. By continuity of X, it is easy to see that the limit
$$\lim_{|\Delta| \to 0} \int_0^t X_s^\Delta \, ds = \lim_{|\Delta| \to 0} \sum_{k=0}^{n-1} X_{t_k}(t_{k+1} - t_k) = \int_0^t X_s \, ds$$
exists almost surely, so $\int_0^t X_s \, ds$ is a well-defined r.v.

1. Show that for any sub–σ-field $\mathcal{G} \subset \mathcal{F}_\infty$, there exists a bounded continuous process $(Y_t)_{t \geq 0}$ such that for every $t \geq 0$, $Y_t = \mathsf{E}[X_t \mid \mathcal{G}]$ a.s.
Hint: define Y_t first for $t \in \mathbb{Q}$ and then consider the extension to $t \in \mathbb{R}$.

2. Show that
$$\mathsf{E}\Big[\int_0^t X_s \, ds \mid \mathcal{G} \Big] = \int_0^t \mathsf{E}[X_s \mid \mathcal{G}] \, ds.$$
Hint: The identity is true for X_t^Δ; then justify the limit $|\Delta| \to 0$ carefully using boundedness and continuity.

3. Let $i = \sqrt{-1}$. For any $\lambda \in \mathbb{R}$, show that

$$e^{i\lambda B_t} + \int_0^t \frac{1}{2}\lambda^2 e^{i\lambda B_s}\, ds, \quad t \geq 0$$

is a martingale.

Note: this implies $f(B_t) - \int_0^t \frac{1}{2} f''(B_s)\, dB_s$ is a martingale if f has a sufficiently nice Fourier transform representation $f(x) = \int e^{i2\pi x\xi} \hat{f}(\xi)\, d\xi$, since $f''(x) = -\int 4\pi^2\xi^2 e^{i\lambda x\xi} \hat{f}(\xi)\, d\xi$.

Exercise 3 Let $t > 0$ and consider the partition $\Delta_n : t_i = it \cdot 2^{-n}$, $0 \leq i \leq 2^n$. Show that

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} (B_{t_{i+1}} - B_{t_i})^2 = t, \quad \text{a.s.}$$

Hint: Compute the L^2-distance, and then use Markov inequality and Borel–Cantelli.