1.(1). The continuity of $W_t(\omega)$ is obvious by definition of $W_t(\omega)$ and continuity of $B_t(\omega)$ and $\tilde{B}_t(\omega)$.

And for $t \geq 0$, $W_t = B_t 1_{\{T \geq t\}} + (B_T + \tilde{B}_t - \tilde{B}_T) 1_{\{T<t\}}$, we know that $B_t, \tilde{B}_t, 1_{\{T<t\}}, 1_{\{T \geq t\}}$ are $\mathcal{F}_t$-measurable.

And since $(B_t)_{t\geq 0}$, $(\tilde{B}_t)_{t \geq 0}$ are progressively measurable, with $T<\infty$ a.s., we obtain that $B_T, \tilde{B}_T \in \mathcal{F}_T$, which implies that $B_T 1_{\{T<t\}}, \tilde{B}_T 1_{\{T<t\}} \in \mathcal{F}_T \cap \{T<t\} \subseteq \mathcal{F}_t$. Then we conclude that $W_t \in \mathcal{F}_t$.

(2). For $m \in \mathbb{N}^*$ and $0 \leq t_1 < \cdots < t_m < +\infty$, $\xi_1, \cdots, \xi_m \in \mathbb{R}$,

$$P(W_{t_1} \leq \xi_1, \cdots, W_{t_m} \leq \xi_m) = \mathbb{E}\left[1_{\{W_{t_1} \leq \xi_1\}} \cdots 1_{\{W_{t_m} \leq \xi_m\}}\right]$$

$$= \mathbb{E}\left[1_{\{W_{t_1} \leq \xi_1\}} \cdots 1_{\{W_{t_m} \leq \xi_m\}} \left(1_{\{T \in [0,t_1)\}} + \sum_{i=1}^{m-1} 1_{\{T \in [t_i, t_{i+1})\}} + 1_{\{T \geq t_m\}}\right)\right]$$

① Firstly note that $\mathbb{E}\left[1_{\{W_{t_1} \leq \xi_1\}} \cdots 1_{\{W_{t_m} \leq \xi_m\}} \cdot 1_{\{T<t_1\}}\right]$

$$= \mathbb{E}\left[1_{\{B_T + \tilde{B}_{t_1} - \tilde{B}_T \leq \xi_1\}} \cdots 1_{\{B_T + \tilde{B}_{t_m} - \tilde{B}_T \leq \xi_m\}} 1_{\{T<t_1\}}\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[1_{\{B_T + \tilde{B}_{t_1} - \tilde{B}_T \leq \xi_1\}} \cdots 1_{\{B_T + \tilde{B}_{t_m} - \tilde{B}_T \leq \xi_m\}} \middle| \mathcal{F}_T\right] 1_{\{T<t_1\}}\right]$$

$$= \mathbb{E}\Big[\mathbb{E}\big[1_{\{\tilde{B}_{t_1} - \tilde{B}_T \le \xi_1 - B_T\}} \cdots 1_{\{\tilde{B}_{t_m} - \tilde{B}_T \le \xi_m - B_T\}} \big| \mathcal{F}_T\big] 1_{\{T<t_1\}}\Big]$$

$$= \mathbb{E}\Big[\mathbb{E}\big[1_{\{B_{t_1} - B_T \le \xi_1 - B_T\}} \cdots 1_{\{B_{t_m} - B_T \le \xi_m - B_T\}} \big| \mathcal{F}_T\big] 1_{\{T<t_1\}}\Big]$$

$$= \mathbb{E}\Big[1_{\{B_{t_1} \le \xi_1\}} \cdots 1_{\{B_{t_m} \le \xi_m\}} \cdot 1_{\{T<t_1\}}\Big].$$

For $i \in \{1, \cdots, m-1\}$,

② $$\mathbb{E}\Big[1_{\{W_{t_1} \le \xi_1\}} \cdots 1_{\{W_{t_m} \le \xi_m\}} 1_{\{T \in [t_i, t_{i+1})\}}\Big]$$

$$= \mathbb{E}\Big[1_{\{B_{t_1} \le \xi_1\}} \cdots 1_{\{B_{t_i} \le \xi_i\}} \cdot 1_{\{B_T + \tilde{B}_{t_{i+1}} - \tilde{B}_T \le \xi_{i+1}\}} \cdots 1_{\{B_T + \tilde{B}_{t_m} - \tilde{B}_T \le \xi_m\}} 1_{\{T \in [t_i, t_{i+1})\}}\Big]$$

$$= \mathbb{E}\Big[1_{\{B_{t_1} \le \xi_1\}} \cdots 1_{\{B_{t_i} \le \xi_i\}} \mathbb{E}\big[1_{\{B_T + \tilde{B}_{t_{i+1}} - \tilde{B}_T \le \xi_{i+1}\}} \cdots 1_{\{B_T + \tilde{B}_{t_m} - \tilde{B}_T\}} \big| \mathcal{F}_T\big] 1_{\{T \in [t_i, t_{i+1})\}}\Big]$$

$$= \mathbb{E}\Big[\mathbb{1}_{\{B_{t_1} \le \xi_1\}} \cdots \mathbb{1}_{\{B_{t_i} \le \xi_i\}} \mathbb{E}\big[\mathbb{1}_{\{\tilde{B}_{t_{i+1}} - \tilde{B}_T \le \xi_{i+1} - B_T\}} \cdots$$

$$\mathbb{1}_{\{\tilde{B}_{t_m} - \tilde{B}_T \le \xi_m - B_T\}} \big| \mathcal{F}_T \big] \mathbb{1}_{\{T \in [t_i, t_{i+1})\}} \Big]$$

$$= \mathbb{E}\Big[\mathbb{1}_{\{B_{t_1} \le \xi_1\}} \cdots \mathbb{1}_{\{B_{t_i} \le \xi_i\}} \cdot \mathbb{E}\big[\mathbb{1}_{\{B_{t_{i+1}} - B_T \le \xi_{i+1} - B_T\}} \cdots$$

$$\mathbb{1}_{\{B_{t_m} - B_T \le \xi_m - B_T\}} \big| \mathcal{F}_T \big] \mathbb{1}_{\{T \in [t_i, t_{i+1})\}} \Big]$$

$$= \mathbb{E}\Big[\mathbb{1}_{\{B_{t_1} \le \xi_1\}} \cdots \mathbb{1}_{\{B_{t_m} \le \xi_m\}} \mathbb{1}_{\{T \in [t_i, t_{i+1})\}} \Big].$$

③ $\mathbb{E}\Big[\mathbb{1}_{\{B_{t_1} \le \xi_1\}} \cdots \mathbb{1}_{\{B_{t_m} \le \xi_m\}} \mathbb{1}_{\{T \ge t_m\}} \Big]$

$$= \mathbb{E}\Big[\mathbb{1}_{\{W_{t_1} \le \xi_1\}} \cdots \mathbb{1}_{\{W_{t_m} \le \xi_m\}} \mathbb{1}_{\{T \ge t_m\}} \Big].$$

Therefore we obtain that $\mathbb{E}\Big[\mathbb{1}_{\{W_{t_1} \le \xi_1\}} \cdots \mathbb{1}_{\{W_{t_m} \le \xi_m\}} \Big]$

$$= \mathbb{E}\Big[\mathbb{1}_{\{B_{t_1} \le \xi_1\}} \cdots \mathbb{1}_{\{B_{t_m} \le \xi_m\}} \Big].$$

Then the proof is done by Monotone class lemma.

2-(1). For notational consistensy, set $T_0 = T_{0a} = \inf\{t \geq 0, B_t = 0\} \equiv 0$. Then $T_a = T_{1a} - T_{0a}$. For $n \in \mathbb{N}^*$, $1 \leq i_1 < \cdots < i_n$, $i_1, \cdots, i_n \in \mathbb{N}^*$, we have:

For $\xi_1, \cdots, \xi_n \geq 0$,

$$P(T_{i_1 a} - T_{(i_1-1)a} < \xi_1, \cdots, T_{i_n a} - T_{(i_n-1)a} < \xi_n)$$

$$= P\left(\bigcap_{k=1}^{n} \{T_{i_k a} - T_{(i_k-1)a} < \xi_k\}\right)$$

$$= P\left(\bigcap_{k=1}^{n} \left\{\sup_{t\in[0,\xi_k]} (B_{T_{(i_k-1)a}+t} - B_{T_{(i_k-1)a}}) > a\right\}\right).$$

For $k \in \{1, \cdots, n\}$, set $E_k := \left\{\sup_{t\in[0,\xi_k]} (B_{T_{(i_k-1)a}+t} - B_{T_{(i_k-1)a}}) > a\right\}$.

We know that by progressively measurablity of $B(\cdot)$, $E_k \in \mathcal{F}_{T_{i_k a}}$, and by Strong Markov Property of $B(\cdot)$, E_k is indepedent of $\mathcal{F}_{T_{(i_k-1)a}}$. Therefore we obtain that

$$P\left(\bigcap_{k=1}^{n} E_k\right) = \mathbb{E}\left[1_{\bigcap_{k=1}^{n} E_k}\right] = \mathbb{E}\left[1_{E_1} \cdots 1_{E_n}\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[1_{E_1} \cdots 1_{E_n} \mid \mathcal{F}_{T_{(i_n-1)a}}\right]\right] = \mathbb{E}\left[1_{E_1} \cdots 1_{E_{n-1}} \cdot \mathbb{E}1_{E_n}\right]$$

$$= \mathbb{E}\left[1_{E_1} \cdots 1_{E_{n-1}}\right] \cdot \mathbb{E}1_{E_n} = \cdots\cdots$$

$$= \mathbb{E}1_{E_1} \cdots \mathbb{E}1_{E_n} = P(E_1) \cdots P(E_n).$$

Then the proof of indepedence is done by monotone class lemma. For the proof of identical distribution of $\{T_{ka} - T_{(k-1)a}\}_{k=1}^{\infty}$, let me show that for $n \in \mathbb{N}^*$, $\mathrm{Law}(T_{na} - T_{(n-1)a}) = \mathrm{Law}(T_a)$.

In fact, for $\xi \geq 0$, $P(T_{na} - T_{(n-1)a} < \xi)$

$$= P\left(\sup_{t \in [0,\xi]} (B_{T_{(n-1)a}+t} - B_{T_{(n-1)a}}) > a\right)$$

$$= P\left(\sup_{t \in [0,\xi]} B_t > a\right)$$ (since $(B_{T+t} - B_T)_{t \geq 0}$ is still a BM for any stopping time $T < \infty$ a.s.)

$$= P(T_a < \xi).$$

Therefore $\{T_{na} - T_{(n-1)a}\}_{n=1}^{\infty}$ is i.i.d. □

(2)

$$e^{-\varphi(\lambda,na)} = \mathbb{E} e^{-\lambda T_{na}} = \mathbb{E} e^{-\lambda(T_a + T_{2a} - T_a + \cdots + T_{na} - T_{(n-1)a})}$$

$$= \mathbb{E}\left[e^{-\lambda T_a} e^{-\lambda(T_{2a}-T_a)} \cdots e^{-\lambda(T_{na}-T_{(n-1)a})}\right]$$

$$\overset{\text{by (1)}}{=} \mathbb{E} e^{-\lambda T_a} \cdot \mathbb{E} e^{-\lambda(T_{2a}-T_a)} \cdots \mathbb{E} e^{-\lambda(T_{na}-T_{(n-1)a})}$$

$= (\mathbb{E} e^{-\lambda T_a})^n$, then we obtain that

$$\varphi(\lambda, na) = -\log e^{-\varphi(\lambda,na)} = -\log(\mathbb{E} e^{-\lambda T_a})^n$$

$$= -n \log \mathbb{E} e^{-\lambda T_a} = -n \log e^{-\varphi(\lambda,a)} = n\varphi(\lambda, a).$$

For $a \in \mathbb{Q}$, $\exists N \in \mathbb{N}^*$, $e \in \mathbb{Q}$ ~~such that~~ with $\frac{1}{e} \in \mathbb{N}^*$ such that $a = Ne$. Then $\varphi(\lambda, 1) = \varphi(\lambda, \frac{1}{e} \cdot e) = \frac{1}{e}\varphi(\lambda, e)$, which means that

$\varphi(\lambda, e) = e\,\varphi(\lambda, 1)$. Also we have that

$$\varphi(\lambda, a) = \varphi(\lambda, Ne) = N\varphi(\lambda, e) = Ne\,\varphi(\lambda, 1) = a\,\varphi(\lambda, 1).$$

Then the equation $\varphi(\lambda, a) = a\,\varphi(\lambda, 1)$ holds for all $a \in (0, +\infty)$ by density of $\mathbb{Q}$ in $\mathbb{R}$ and continuity of Laplace Transform together with the fact that $T_{a_n} \to T_a$ weakly (in distribution) if $a_n \to a$ as $n \to +\infty$.

(3). For $\xi > 0$, $P(T_{a\lambda} < \xi) = P(\sup_{t\in[0,\xi]} B_t > a\lambda)$

$= P(\sup_{t\in[0,\xi]} \lambda B_{\frac{t}{\lambda^2}} > a\lambda)$

$= P(\sup_{t\in[0,\xi]} B_{\frac{t}{\lambda^2}} > a) = P(\sup_{\frac{t}{\lambda^2}\in[0,\frac{\xi}{\lambda^2}]} B_{\frac{t}{\lambda^2}} > a)$

$= P(T_a < \frac{\xi}{\lambda^2}) = P(\lambda^2 T_a < \xi)$, which implies that $T_{a\lambda}$ has the same distribution as $\lambda^2 T_a$.

(4). By (3) we know that $\mathrm{Law}(T_{a\lambda}) = \mathrm{Law}(\lambda^2 T_a)$ for all $a, \lambda > 0$, which implies that $\mathbb{E} e^{-\lambda^2 T_a} = \mathbb{E} e^{-T_{a\lambda}}$, which means that $\varphi(\lambda^2, a) = \varphi(1, \lambda a)$.

It implies that for $a, \lambda > 0$, $\cancel{\mathbb{E} e^{-\lambda T_a} = \mathbb{E}}$

$$\cancel{e^{-\varphi(\lambda, a)} = \mathbb{E} e^{-\lambda T_a} = \mathbb{E} e^{-T_{\sqrt{\lambda} a}} = e^{-\varphi(1,}}$$

$$\mathbb{E} e^{-\lambda T_a} = e^{-\varphi(\lambda, a)} = e^{-\varphi(1, \sqrt{\lambda} a)} = e^{-\sqrt{\lambda} a \varphi(1,1)}.$$

Then the proof is done by letting $c = \varphi(1,1)$.

3. Suppose $|\xi_i| \leq C_i$, $i \in \mathbb{N}^*$.

(1). For $t \in [0,+\infty)$. Since $t_i \to +\infty$ increasingly as $i \to +\infty$, then $\exists\, n(t) \in \mathbb{N}^*$ such that $t_{n(t)-1} < t$ and $t_{n(t)} \geq t$.

Then we have $M_t = \sum_{i=0}^{\infty} \xi_i (B_{t_{i+1}\wedge t} - B_{t_i \wedge t})$

$= \sum_{i=0}^{n(t)-1} \xi_i (B_{t_{i+1}\wedge t} - B_{t_i\wedge t})$.

① Firstly $\mathbb{E}|M_t| \leq \mathbb{E}\sum_{i=0}^{n(t)-1} |\xi_i||B_{t_{i+1}\wedge t} - B_{t_i\wedge t}|$

$\leq \sum_{i=0}^{n(t)-1} C_i \mathbb{E}|B_{t_{i+1}\wedge t} - B_{t_i\wedge t}| < \infty$.

② Since $\xi_i \in \mathcal{F}_{t_i}$, then for $i \in \{0,1,\cdots,n(t)-1\}$, $t_i \leq t_{n(t)-1} < t$, thus $\xi_i \in \mathcal{F}_t$ for any $i \in \{0,1,\cdots,n(t)-1\}$. And obviously $|B_{t_{i+1}\wedge t} - B_{t_i\wedge t}| \in \mathcal{F}_t$. Therefore $M_t \in \mathcal{F}_t$.

③. For $0 \leq s < t$, $\exists\, n(s) \in \mathbb{N}^*$ such that $s \in (t_{n(s)-1}, t_{n(s)}]$, and $n(s) \leq n(t)$. Firstly if $n(s) < n(t)$, then $\mathbb{E}[M_t | \mathcal{F}_s]$

$= \mathbb{E}\left[\sum_{i=0}^{n(t)-1} \xi_i (B_{t_{i+1}\wedge t} - B_{t_i\wedge t}) \,\Big|\, \mathcal{F}_s\right] = \mathbb{E}\left[\sum_{i=0}^{n(s)-1} \xi_i (B_{t_{i+1}\wedge t} - B_{t_i\wedge t})\right.$

$\left. + \sum_{i=n(s)}^{n(t)-1} \xi_i (B_{t_{i+1}\wedge t} - B_{t_i\wedge t}) \,\Big|\, \mathcal{F}_s\right] = \sum_{i=0}^{n(s)-1} \xi_i (B_{t_{i+1}\wedge s} - B_{t_i\wedge s})$

$+ \mathbb{E}\left[\sum_{i=n(s)}^{n(t)-1} \xi_i (B_{t_{i+1}\wedge t} - B_{t_i\wedge t}) \,\Big|\, \mathcal{F}_s\right]$

$$= M_s + \sum_{i=n(s)}^{n(t)-1} \mathbb{E}\Big[\mathbb{E}\big[\xi_i (B_{t_{i+1}\wedge t} - B_{t_i\wedge t}) \mid \mathcal{F}_{t_i}\big] \Big| \mathcal{F}_s\Big]$$

$$= M_s + \sum_{i=n(s)}^{n(t)-1} \mathbb{E}\big[\xi_i \cdot 0 \mid \mathcal{F}_s\big] = M_s .$$

If $n(s) = n(t)$, then $\mathbb{E}[M_t \mid \mathcal{F}_s] = \mathbb{E}\Big[\sum_{i=0}^{n(s)-1} \xi_i (B_{t_{i+1}\wedge t} - B_{t_i\wedge t})$ $\Big|\mathcal{F}_s\Big] = \mathbb{E}\Big[\sum_{i<n(s)-1} \xi_i (B_{t_{i+1}\wedge t} - B_{t_i\wedge t}) + \xi_{n(s)-1}(B_{t_{n(s)}\wedge t} - B_{t_{n(s)-1}\wedge t})$ $\Big|\mathcal{F}_s\Big] = \sum_{i<n(s)-1} \xi_i (B_{t_{i+1}\wedge t} - B_{t_i\wedge t}) + \xi_{n(s)-1}\mathbb{E}\big[B_{t_{n(s)}\wedge t}$ $- B_{t_{n(s)-1}\wedge t} \mid \mathcal{F}_s\big] = \sum_{i<n(s)-1} \xi_i (B_{t_{i+1}\wedge s} - B_{t_i\wedge s}) + \xi_{n(s)-i}$ $(B_{t(s)\wedge s} - B_{t_{n(s)-1}\wedge s}) = M_s$.

Therefore $(M_t)_{t\geq 0}$ is a martingale .

(2). For $0 \leq i < j$, $t \geq 0$, $\mathbb{E}\big[\xi_i (B_{t_{i+1}\wedge t} - B_{t_i\wedge t}) \cdot \xi_j \cdot (B_{t_{j+1}\wedge t} - B_{t_j\wedge t})\big]$

$$= \mathbb{E}\Big[\mathbb{E}\big[\xi_i \xi_j (B_{t_{i+1}\wedge t} - B_{t_i\wedge t}) \cdot (B_{t_{j+1}\wedge t} - B_{t_j\wedge t}) \mid \mathcal{F}_{t_j}\big]\Big]$$

$$= \mathbb{E}\big[\xi_i \xi_j (B_{t_{i+1}\wedge t} - B_{t_i\wedge t}) \cdot 0\big] = 0 .$$

Thus $\mathbb{E}|M_t|^2 = \mathbb{E}\Big[\Big(\sum_{i=0}^{n(t)-1} \xi_i (B_{t_{i+1}\wedge t} - B_{t_i\wedge t})\Big)^2\Big]$

$$= \sum_{i=0}^{n(t)-1} \mathbb{E}\big[|\xi_i|^2 \cdot |B_{t_{i+1}\wedge t} - B_{t_i\wedge t}|^2\big]$$

$$= \sum_{i=0}^{n(t)-1} \mathbb{E}\Big[\mathbb{E}\big[|\xi_i|^2 |B_{t_{i+1}\wedge t} - B_{t_i\wedge t}|^2 \mid \mathcal{F}_{t_i}\big]\Big]$$

$$= \sum_{i=0}^{n(t)-1} \mathbb{E}\left[|\xi_i|^2 \cdot \mathbb{E}|B_{t_{i+1}\wedge t} - B_{t_i \wedge t}|^2 \right]$$

$$= \sum_{i=0}^{n(t)-1} \mathbb{E}|\xi_i|^2 \cdot (t_{i+1}\wedge t - t_i \wedge t)$$

$$= \sum_{i=0}^{\infty} (t_{i+1}\wedge t - t_i\wedge t) \cdot \mathbb{E}|\xi_i|^2 .$$