HW4

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Exercise 1 Let $B = (B_t)_{t\geq 0}$ and $\tilde{B} = (\tilde{B}_t)_{t\geq 0}$ be two independent $(\mathcal{F}_t)$-adapted Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$. Let T be an a.s. finite stopping time. Define

$$W_t(\omega) = \begin{cases} B_t(\omega), & t \leq T(\omega), \\ B_{T(\omega)} + \Big(\tilde{B}_t(\omega) - \tilde{B}_{T(\omega)}(\omega)\Big), & t > T(\omega). \end{cases}$$

1. Show that $(W_t)_{t\geq 0}$ is a continuous, $(\mathcal{F}_t)$-adapted stochastic process.
2. Prove that $W = (W_t)_{t\geq 0}$ is a standard Brownian motion by showing that W and B have the same finite dimensional distribution, namely, for all $0 \leq t_1 < t_2 < \cdots < t_m$ and all Borel sets $A_1, A_2, \ldots, A_m$,
$$\mathsf{P}(W_{t_1} \in A_1, \cdots, W_{t_m} \in A_m) = \mathsf{P}(B_{t_1} \in A_1, \cdots, B_{t_m} \in A_m).$$

Exercise 2 Let $B = (B_t)_{t\geq 0}$ be the standard Brownian motion. For $a > 0$, let $T_a = \inf\{t \geq 0 : B_t = a\}$ be the first hitting time of a. For $\lambda > 0$, define the Laplace transform of T_a: $e^{-\varphi(\lambda,a)} = \mathsf{E}e^{-\lambda T_a}$. It is not hard to show that φ is a continuous function and we will assume that.

1. Use the strong Markov property to show that $T_a, T_{2a} - T_a, T_{3a} - T_{2a}, \ldots$ are i.i.d. random variables.
2. Show that
$$\varphi(\lambda, na) = n\varphi(\lambda, a), \quad n \geq 1,$$
and use continuity to conclude that $\varphi(\lambda, a) = \varphi(\lambda, 1)a$, $a > 0$.
3. Use the fact that $(\lambda B_{\lambda^{-2}t})_{t\geq 0}$ is also a standard Brownian motion for every $\lambda > 0$ to show that $T_{a\lambda}$ and $\lambda^2 T_a$ have the same distribution.
4. Show that $\varphi(\lambda^2, a) = \varphi(1, \lambda a)$ and conclude that there is a constant $c > 0$ such that
$$\mathsf{E}e^{-\lambda T_a} = e^{-c\sqrt{\lambda}a}.$$

Exercise 3 Let $(B_t)_{t\geq 0}$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$. Let $0 = t_0, t_1, \cdots$ be a sequence increasing to ∞ and let $(\xi_i)_{i\geq 0}$ be a sequence of bounded r.v.'s such that $\xi_i \in \mathcal{F}_{t_i}$. Define

$$M_t = \sum_{i=0}^{\infty} \xi_i \big(B_{(t_{i+1}\wedge t)} - B_{(t_i \wedge t)}\big).$$

1. Show that $(M_t)_{t\geq 0}$ is a martingale.
   Hint: it suffices to show that $\mathsf{E}[M_t \mid \mathcal{F}_s] = M_s$ *for all* $s < t$ *in every interval* $[t_i, t_{i+1}]$.
2. Show that
$$\mathsf{E}M_t^2 = \sum_{i=0}^{\infty} (t_{i+1} \wedge t - t_i \wedge t) \cdot \mathsf{E}|\xi_i|^2.$$