1. Denote $A_{M,n} := \left\{ \sup_{t\in(0,\frac{1}{n}]} \frac{B_t}{\sqrt{t}} > M \right\}$, $n \in \mathbb{N}^*$.

$$A_M = \bigcap_{n=1}^{\infty} A_{M,n}.$$

(1). By monotone continuity of probability, we have

$$P(A_M) = P\left(\bigcap_{n=1}^{\infty} A_{M,n}\right) = \lim_{n\to+\infty} P(A_{M,n})$$

$$\geqslant \lim_{n\to+\infty} P\left(\left\{\frac{B_{\frac{1}{n}}}{\sqrt{\frac{1}{n}}} > M\right\}\right) = \lim_{n\to+\infty} P(N(0,1) > M)$$

$$= P(N(0,1) > M) = P(N(0,1) \geqslant M).$$

(2). Denote by $(\mathcal{F}_t^B)_{t\geqslant 0}$ the natural filtration of $B(\cdot)$. Then by definition we have that $A_{M,n} \in \mathcal{F}^B_{\frac{1}{n}}$ for $n \in \mathbb{N}^*$, which implies that $A_M = \bigcap_{n=1}^{\infty} A_{M,n} \in \bigcap_{n=1}^{\infty} \mathcal{F}^B_{\frac{1}{n}} = \mathcal{F}^B_{0+}$. By 0-1 Law together with the fact that $P(A_M) = P(N(0,1) \geqslant M) > 0$, we obtain that $P(A_M) = 1$.

(3). It is a direct consequence of the fact that $P(A_M) = 1$ Since $w \in A_M \Leftrightarrow \sup_{t\in(0,\frac{1}{n}]} \frac{B_t(w)}{\sqrt{t}} > M$ for all $n \in \mathbb{N}^*$.

(4). Note that $P\left(\bigcap_{M=1}^{\infty} A_M\right) = \lim_{M\to+\infty} P(A_M) = 1$ and the fact that $w \in \bigcap_{M=1}^{\infty} A_M \Leftrightarrow$ For $\forall n \in \mathbb{N}^*$, $\sup_{t\in(0,\frac{1}{n}]} \frac{B_t(w)}{\sqrt{t}} = +\infty$. The proof is therefore completed.

2. (1). For $t\in[0,1]$, $\mathbb{E}X_t = \mathbb{E}[B_t - tB_1] = \mathbb{E}B_t - t\mathbb{E}B_1 = 0$.

For $n\in\mathbb{N}^*$, and $0\leqslant t_1<\cdots<t_n\leqslant 1$, $(c_1,\cdots,c_n)\in\mathbb{R}^n$, we have

$\sum_{i=1}^{n} c_i X_{t_i} = \sum_{i=1}^{n} c_i(B_{t_i} - t_iB_1) = \sum_{i=1}^{n} c_i B_{t_i} - (\sum_{i=1}^{n} t_i)B_1$, which

is Gaussian, since $B(\cdot)$ is a Gaussian process.

Thus $X(\cdot)$ is a centered Gaussian process.

And for $0\leqslant s<t\leqslant 1$, $\mathbb{E}[X_sX_t] = \mathbb{E}[(B_s - sB_1)(B_t - tB_1)]$

$= \mathbb{E}[B_sB_t - B_s\cdot tB_1 - sB_1B_t + stB_1^2]$

$= s - ts - st + st = s(1-t)$.

(2). For $t>s>s_1>\cdots>s_n\geqslant 0$, and $i\in\{1,2,\cdots,n\}$,

$\mathbb{E}[(X_t - \frac{1-t}{1-s}X_s)X_{s_i}] = \mathbb{E}X_tX_{s_i} - \frac{1-t}{1-s}\mathbb{E}X_sX_{s_i}$

$= \cancel{s_i(1-s)}\ s_i(1-t) - \frac{1-t}{1-s}s_i(1-s) = 0$. Since $X(\cdot)$ is Gaussian,

then we have $X_t - \frac{1-t}{1-s}X_s$ is indepedent from $\bigcup_{i=1}^{n}\sigma(X_{s_i})$,

and since the latter is a π-system, we obtain that

$X_t - \frac{1-t}{1-s}X_s$ is indepedent of $\sigma(X_{s_1},\cdots X_{s_n})$ by ~~monotone~~ monotone class lemma

(3). By (2) we know that for $t > s \in [0,1)$, $X_t - \frac{1-t}{1-s} X_s$ is indepedent of the collection

$$\mathcal{G} := \left\{ \bigcap_{i=1}^{n} \{ X_{s_i} \in B_i \} \right\}_{\substack{0 \leq s_1 < \cdots < s_n \leq s, \\ n \in \mathbb{N}^*}} .$$

By the fact that $\mathcal{G}$ is a π-system and $\mathcal{F}_s^X = \sigma(\mathcal{G})$, we obtain that $X_t - \frac{1-t}{1-s} X_s$ is indepedent of $\mathcal{F}_s^X$. By the Monotone class lemma.

(4)

For $0 \leq s < t \leq 1$, and any bounded measurable function $F(\cdot)$, we could find a bounded measurable function $G: \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto F(x+y)$. And we have:

$$\cancel{\mathbb{E}[F(X_t) | \mathcal{F}_s^X] = \mathbb{E}[F(X_t - X_s + X_s) | \mathcal{F}_s^X]}$$

$$\cancel{= \mathbb{E}[G(X_t - X_s, X_s)}$$

$$\mathbb{E}[F(X_t) | \mathcal{F}_s^X] = \mathbb{E}\left[F\left(X_t - \frac{1-t}{1-s} X_s + \frac{1-t}{1-s} X_s\right) \Big| \mathcal{F}_s^X\right]$$

$$= \mathbb{E}\left[G\left(X_t - \frac{1-t}{1-s} X_s, \frac{1-t}{1-s} X_s\right) \Big| \mathcal{F}_s^X\right] = \mathbb{E}\left[G\left(X_t - \frac{1-t}{1-s} X_s, y\right)\right]\Big|_{y = \frac{1-t}{1-s} X_s}$$

Which is $\sigma(X_s)$ measurable.

Therefore $\mathbb{E}\left[F(X_t) \mid \mathcal{F}_s^X\right] = \mathbb{E}\left[\mathbb{E}\left[F(X_t) \mid \mathcal{F}_s^X\right] \Big| X_s\right]$

$$= \mathbb{E}\left[F(X_t) \mid X_s\right].$$

The proof is done.