HW3

March 5, 2024

Exercise 1 For $M > 0$, define $A_M = \bigcap_{n\geq 1}\{ \sup_{0<t\leq 1/n} \frac{B_t}{\sqrt{t}} > M\}$.

1. Show that $\mathsf{P}(A_M) \geq \mathsf{P}(\mathcal{N}(0,1) \geq M)$.
2. Use the zero-one law to deduce that $\mathsf{P}(A_M) = 1$.
3. For every $M > 0$, show that with probability one,
$$\sup_{0<t\leq \frac{1}{n}} \frac{B_t}{\sqrt{t}} > M, \quad \forall n \geq 1.$$
4. Show that with probability one,
$$\sup_{0<t\leq \frac{1}{n}} \frac{B_t}{\sqrt{t}} = +\infty, \quad \forall n \geq 1.$$

Exercise 2 Let $(B_t)_{t\in[0,1]}$ be the Brownian motion and define $X_t = B_t - tB_1$, $t \in [0,1]$. The process $X = (X_t)_{t\in[0,1]}$ is called the "Brownian Bridge".

1. Show that $(X_t)_{t\geq 0}$ is a centered Gaussian process with covariance
$$\mathsf{E}X_tX_s = s(1-t), \quad \forall 0 \leq s < t \leq 1.$$
2. Let $t > s > s_1 > s_2 > \cdots > s_n \geq 0$. Show that
$$\mathsf{E}\Big(X_t - \frac{1-t}{1-s}X_s\Big)X_{s_i} = 0, \quad 1 \leq i \leq n.$$
Deduce that $X_t - \frac{1-t}{1-s}X_s$ is independent of $(X_{s_1}, \ldots, X_{s_n})$.
3. Let $t > s$. Show that $X_t - \frac{1-t}{1-s}X_s$ is independent of $\mathcal{F}_s^X$.
4. Show that $(X_t)_{t\in[0,1]}$ is Markov.