1. Firstly let us verify that the distribution:

$$\binom{X}{Y} \sim N(0, I_{2\times 2})$$

satisfies the rotational invariance on $\mathbb{R}^2$.

Let $A \in \mathbb{R}^{2\times 2}$ be an orthonormal matrix, i.e., $A^TA = AA^T = I$

For $u \in \mathbb{R}^2$, we have $\varphi_{(X,Y)^T}(u) = \exp\{-\frac{1}{2}\|u\|^2\}$.

And
$$\begin{aligned}\varphi_{A\binom{X}{Y}}(u) &= \exp\{-\tfrac{1}{2}\langle A^Tu, IA^Tu\rangle\} \\ &= \exp\{-\tfrac{1}{2}\langle u, AIA^Tu\rangle\} \\ &= \exp\{-\tfrac{1}{2}\langle u, u\rangle\} \\ &= \varphi_{(X,Y)^T}(u).\end{aligned}$$

Then it suffices to show the uniqueness of rotation-invariant probability on $\mathbb{R}^2$. It could be done by noticing that the set of rotation-invariant measures ~~is~~ are (left) Haar measures under the topological group of rotations on $\mathbb{R}^2$. (And since obviously these measures must be absolutely continuous w.r.t. Lesbeague measure, then they are Radon measures.) Thus by the uniqueness of left-Haar measures, the proof is done.

2. $\int_0^1\int_0^1 s\wedge t\,\lambda(ds)\,\lambda(dt)$

$$= \int_0^1\left(\int_0^t s\,\lambda(ds) + \int_t^1 t\,\lambda(ds)\right)\lambda(dt)$$

$$= \int_0^1\int_0^t s\,\lambda(ds)\,\lambda(dt) + \int_0^1\int_t^1 t\,\lambda(ds)\,\lambda(dt)$$

$$= \int_0^1\int_0^t s\,d(-f(s))\,\lambda(dt) + \int_0^1 t\int_t^1 d(-f(s))\,\lambda(dt)$$

$$= \int_0^1\left(-sf(s)\Big|_0^t + \int_0^t f(s)\,ds\right)\lambda(dt) + \int_0^1 t\,(f(t)-f(1))\,\lambda(dt)$$

$$= \int_0^1\int_0^t f(s)\,ds\,\lambda(dt)$$

$$= \int_0^1\int_0^t f(s)\,ds\,d(-f(t))$$

$$= -f(t)\cdot\int_0^t f(s)\,ds\,\Big|_0^1 + \int_0^1 f(t)\cdot f(t)\,dt$$

$$= 0 + \int_0^1 |f(t)|^2\,dt$$

3.

(1). By condition, for fixed $\bar{s}\in[0,1]$ or $\bar{t}\in[0,1]$, we have that $(B_{\bar{s},t})_{t\in[0,1]}$ and $(B_{s,\bar{t}})_{s\in[0,1]}$ are (pre-) Standard Brownian motions. Then we have that for $p\geq 1$, $\exists\, \widetilde{K}_p>0$ such that for $s,t,s',t'\in[0,1]$,

$$\mathbb{E}|B_{\bar{s},t}-B_{\bar{s},t'}|^{2p}\leq \widetilde{K}_p|t-t'|^p,$$

$$\mathbb{E}|B_{s,\bar{t}}-B_{s',\bar{t}}|^{2p}\leq \widetilde{K}_p|s-s'|^p.$$

Therefore we have that

$$\mathbb{E}|B_{s,t}-B_{s',t'}|^{2p}=\mathbb{E}\left[|B_{s,t}-B_{s,t'}+B_{s,t'}-B_{s',t'}|^{2p}\right]$$

$$\leq\mathbb{E}\left[(|B_{s,t}-B_{s,t'}|+|B_{s,t'}-B_{s',t'}|)^{2p}\right]$$

$$\leq\mathbb{E}\left[2^{2p}(|B_{s,t}-B_{s,t'}|^{2p}+|B_{s,t'}-B_{s',t'}|^{2p})\right]$$

$$\leq 2^{2p}\widetilde{K}_p\left(|t-t'|^p+|s-s'|^p\right).$$

(2). For $\gamma\in(0,\frac{1}{2})$, we could find $p\geq 1$ sufficiently large such that $0<\frac{1}{p}<\frac{1}{2}-\gamma$ holds, which implies that $p-2p\gamma-2>0$. By Markov's inequality, we have for any $n\in\mathbb{N}^*$, $0\leq k,k',l,l'\leq 2^n$, $|k-k'|+|l-l'|\leq 1$,

$$P\left(|B_{\frac{k}{2^n},\frac{l}{2^n}}-B_{\frac{k'}{2^n},\frac{l'}{2^n}}|>2^{-\gamma n}\right)=P\left(|B_{\frac{k}{2^n},\frac{l}{2^n}}-B_{\frac{k'}{2^n},\frac{l'}{2^n}}|^{2p}>2^{-2p\gamma n}\right.$$

$$\leq \frac{\mathbb{E}\left|B_{\frac{k}{2^n},\frac{l}{2^n}} - B_{\frac{k'}{2^n},\frac{l'}{2^n}}\right|^{2p}}{2^{-2p\gamma n}}$$

$$\leq \frac{K_p\left(\left|\frac{k}{2^n}-\frac{k'}{2^n}\right|^p + \left|\frac{l}{2^n}-\frac{l'}{2^n}\right|^p\right)}{2^{-2p\gamma n}}$$

$$\leq K_p \cdot 2^{-np} \cdot 2^{-(-2p\gamma n)} = K_p \cdot 2^{-n(p-2p\gamma)}.$$

Then we have $P\left(\sup_{\substack{0\leq k,k'\leq 2^n \\ l,l'}} \left|B_{\frac{k}{2^n},\frac{l}{2^n}} - B_{\frac{k'}{2^n},\frac{l'}{2^n}}\right| > 2^{-\gamma n}\right)$

$$\leq 2^n \cdot 2^n \cdot K_p \cdot 2^{-n(p-2p\gamma)}$$

$$= K_p \cdot 2^{-n(p-2p\gamma-2)}.$$

Since $p-2p\gamma-2>0$, we obtain that $\sum_{n=1}^{\infty} 2^{-(p-2p\gamma-2)n} < \infty$.

By Borel-Cantelli Theorem, we have that

$$P\left(\liminf_{n\to+\infty}\left\{\sup_{\substack{0\leq k,k',l,l'\leq 2^n \\ |k-k'|+|l-l'|\leq 1}} \left|B_{\frac{k}{2^n},\frac{l}{2^n}} - B_{\frac{k'}{2^n},\frac{l'}{2^n}}\right| \leq 2^{-\gamma n}\right\}\right) = 1,$$

which completes the proof of the desired results.

4. Set $E_{n,k} = \left[\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}\right)$,

$$F_{n,k} = \left[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}\right), \quad n\in \mathbb{N},\ 0\le k\le 2^n-1$$

(1). $\int_0^1 e_{n,k}(x)\, e_{n',k'}(x)\, dx$

$$= \int_0^1 2^{\frac{n}{2}}(1_{E_{n,k}} - 1_{F_{n,k}}) \cdot 2^{\frac{n'}{2}}(1_{E_{n',k'}} - 1_{F_{n',k'}})\, dx$$

$$= 2^{\frac{1}{2}(n+n')} \cdot \int_0^1 \Big(1_{E_{n,k}\cap E_{n',k'}}(x) + 1_{F_{n,k}\cap F_{n',k'}}(x) - 1_{E_{n,k}\cap F_{n',k'}}(x) - 1_{E_{n',k'}\cap F_{n,k}}(x)\Big)\, dx \quad (*)$$

~~① For $n=n'$, $k=k'$, we have $E_{n,k}\cap E_{n',k'} = E_{n,k}$, $E_{n,k}\cap F_{n,k} = \phi$. Therefore $(*) = 2^{\frac{1}{2}\cdot 2n}\cdot 2\cdot \frac{1}{2^{n+1}} = 1$.~~

~~② For $n=n'$, $k\neq k'$,~~

① For $n=n'$, $k=k'$, we have $|E_{n,k}\cap E_{n',k'}| = |F_{n,k}\cap F_{n',k'}| = \frac{1}{2^{n+1}}$ and $|E_{n,k}\cap F_{n',k'}| = |E_{n',k'}\cap F_{n,k}| = 0$, thus we obtain that $(*) = 2^{\frac{1}{2}\cdot 2n}\cdot 2\cdot \frac{1}{2^{n+1}} = 1$

② For $n=n'$, $k\neq k'$, we have $\phi = E_{n,k}\cap E_{n',k'} = F_{n,k}\cap F_{n',k}$ and $\phi = E_{n,k}\cap F_{n',k'} = E_{n',k'}\cap F_{n,k}$ (by odd and even difference)

Which implies $(*) = 0$.

③ For $n \neq n'$, by some observation we could obtain that

$$E_{n,k} \cap E_{n',k'} = \phi \Leftrightarrow F_{n,k} \cap F_{n',k'} = \phi \Leftrightarrow E_{n,k} \cap F_{n',k'} = \phi$$

$$\Leftrightarrow E_{n',k'} \cap F_{n,k} = \phi .$$

And when ~~they~~ one of these pairs intersects, we have that all their length are the same, which are equal to $\frac{1}{2^{\bar{n}+1}}$, where $\bar{n} = \max(n, n')$.

Thus $(*) = 0$ in this case.

In conclude we have $\int_0^1 e_{n,k}(x)\, e_{n',k'}(x)\, dx = 1_{n=n'} 1_{k=k'}$.

(2) $\cdot$ $$\sup_{t\in[0,1]} |\Delta B_t^n| = \sup_{t\in[0,1]} \left| \sum_{k=0}^{2^n-1} \xi_{n,k} \int_0^t e_{n,k}(x)\, dx \right|$$

$$= \sup_{t\in[0,1]} \left| \sum_{k=0}^{2^n-1} \xi_{n,k} \left(2^{-(n+1)} \cdot \left(x - \frac{k}{2^n}\right) 1_{E_{n,k}} - 2^{-(n+1)} \left(x - \frac{k+1}{2^n}\right) 1_{F_{n,k}} \right) \right| 2^{\frac{n}{2}}$$

$$\leq \sup_{t\in[0,1]} \left| \max_{0\leq k\leq 2^n-1} |\xi_{n,k}| \cdot \left(\sum_{k=0}^{2^n-1} 1_{E_{n,k}} + 1_{F_{n,k}} \right) \right| \cdot 2^{-\frac{n}{2}-1}$$

$$\leq \max_{0\leq k\leq 2^n-1} |\xi_{n,k}| \cdot 2^{-\frac{n}{2}-1}$$

$$\leq \max_{0\leq k\leq 2^n-1} |\xi_{n,k}| \cdot 2^{-\frac{1}{2}n} .$$

(3). For each $n \in \mathbb{N}^*$, $0 \leq k \leq 2^n - 1$, we have

$$P(|\xi_{n,k}| > n) \leq e^{-\frac{1}{2}n^2}.$$

Then $P(\sup_{0 \leq k \leq 2^n-1} |\xi_{n,k}| > n)$

$$= P\left(\bigcup_{k=0}^{2^n-1} \{|\xi_{n,k}| > n\}\right)$$

$$\leq \sum_{k=0}^{2^n-1} P(|\xi_{n,k}| > n)$$

$$\leq 2^n \cdot e^{-\frac{1}{2}n^2}.$$

Since $\sum_{n=1}^{\infty} 2^n \cdot e^{-\frac{1}{2}n^2} < \infty$,

(By root test: $\lim_{n \to +\infty} \left(\frac{2^n}{e^{\frac{1}{2}n^2}}\right)^{\frac{1}{n}} = \lim_{n \to +\infty} \frac{2}{e^{\frac{1}{2}n}} = 0 < 1$)

then by Borel- ~~Cantelli~~ Cantelli Theorem we have

$$P\left(\liminf_{n \to +\infty} \{\sup_{0 \leq k \leq 2^n-1} |\xi_{n,k}| \leq n\}\right) = 1,$$

which is exactly that: For a.s. $\omega \in \Omega$, there exists $n_0 = n_0(\omega) \in \mathbb{N}^*$ such that for all $n > n_0$, $0 \leq k \leq 2^n - 1$,

$$|\xi_{n,k}| \leq n.$$

Fix $\varepsilon > 0$.

(4). For ~~a.s.~~ all $\omega \in \Omega$, we have that for all $N < N'$, $N, N' \in \mathbb{N}^*$, it holds that:

$$\sup_{t\in[0,1]} |B_t^N(\omega) - B_t^{N'}(\omega)| = \sup_{t\in[0,1]} \Big|\sum_{n=N+1}^{N'} \Delta B_t^n(\omega)\Big|$$

$$\leq \sum_{n=N+1}^{N'} \sup_{t\in[0,1]} |\Delta B_t^n(\omega)|$$

$$\leq \sum_{n=N+1}^{N'} 2^{-\frac{1}{2}n} \cdot \max_{0\leq k\leq 2^n-1} |\xi_{n,k}| \qquad (*)$$

Then by (3), we have that for a.s. $\omega \in \Omega$, there exists $N_1(\omega) > 0$ such that for $N, N' \geq N_1$, it holds that

$$(*) \leq \sum_{n=N+1}^{N'} 2^{-\frac{1}{2}n} \cdot n .$$

And by the fact $\sum_{n=1}^{\infty} n \cdot 2^{-\frac{1}{2}n} < \infty$ (By root test: $\lim_{n\to+\infty} \left(\frac{n}{2^{\frac{1}{2}n}}\right)^{\frac{1}{n}} = \frac{1}{\sqrt{2}} < 1$).

we have that there exists $N_2 > 0$ such that for $N, N' > N_2$,

$$\sum_{n=N+1}^{N'} 2^{-\frac{1}{2}n} \cdot n < \varepsilon .$$

Therefore for $N, N' > \max(N_1, N_2)$, we have

$$\sup_{t\in[0,1]} |B_t^N(\omega) - B_t^{N'}(\omega)| < \varepsilon .$$

The proof is done.