HW2

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Exercise 1 Let X and Y be i.i.d. with $\mathsf{E}X = \mathsf{E}Y = 0$ and $\mathsf{E}X^2 = \mathsf{E}Y^2 = 1$. Suppose that the distribution of (X, Y) is rotational invariant, i.e.,

$$\mathcal{L}(X, Y) = \mathcal{L}(X\cos\theta + Y\sin\theta, -X\sin\theta + Y\cos\theta), \quad \forall\theta \in \mathbb{R}.$$

Show that $\mathcal{L}(X) = \mathcal{L}(Y) = \mathcal{N}(0, 1)$.

Hint: rotational invariance implies that the ch.f. takes the form $\varphi_{X,Y}(\xi, \eta) = F(\xi^2 + \eta^2)$.

Exercise 2 Let $f(t) = \lambda((t, 1])$.

1. Suppose that $\lambda(dt) = \rho(t)\,dt$ for some $\rho \in \mathcal{C}[0, 1]$. Show that

$$\int_0^1 \int_0^1 (s \wedge t)\,\lambda(ds)\lambda(dt) = \int_0^1 |f(t)|^2\,dt.$$

   Hint: use integration by parts.

2. (Optional) Prove the same identity for an arbitrary signed measure $\lambda(dt)$.

   Hint: if $\lambda(dt)$ *is a signed measure, then* f *defined as above has bounded variation and* $\lambda(dt) = d\big(-f(t)\big)$. *Use integration by parts for Riemann–Stieltjes integrals.*

Exercise 3 The *Brown sheet* $(\mathbb{B}_{s,t})_{s,t\in[0,1]}$ is a centered Gaussian process with covariance

$$\mathsf{E}\mathbb{B}_{s,t}\mathbb{B}_{s',t'} = (s \wedge s')(t \wedge t'), \quad s, t, s', t' \in [0, 1].$$

It can be constructed via GWN with $H = L^2([0, 1]^2, \mathcal{B}([0, 1]^2), ds \times dt)$ and $\mathbb{B}_{s,t} = G\big(\mathbb{1}_{[0,s]\times[0,t]}\big)$.

1. Show that for each $p \geq 1$, there is some constant $K_p > 0$,

$$\mathsf{E}|\mathbb{B}_{s,t} - \mathbb{B}_{s',t'}|^{2p} \leq K_p\big(|s - s'|^p + |t - t'|^p\big), \quad s, t, s', t' \in [0, 1].$$

2. Let $0 < \gamma < 1/2$. Show that with probability one, there is a random constant $n_0 = n_0(\omega)$ such that for all $n \geq n_0$,

$$\Big|\mathbb{B}_{\frac{k}{2^n},\frac{\ell}{2^n}} - \mathbb{B}_{\frac{k'}{2^n},\frac{\ell'}{2^n}}\Big| \leq 2^{-\gamma n}, \quad 0 \leq k, \ell, k', \ell' \leq 2^n,\ |k - k'| + |\ell - \ell'| \leq 1.$$

Exercise 4 For $n \geq 0$ and $0 \leq k \leq 2^n - 1$, let

$$e_{n,k}(x) = \begin{cases} 2^{\frac{n}{2}}, & \dfrac{k}{2^n} \leq x < \dfrac{2k+1}{2^{n+1}}, \\ -2^{\frac{n}{2}}, & \dfrac{2k+1}{2^{n+1}} \leq x < \dfrac{k+1}{2^n}, \\ 0, & \text{otherwise}, \end{cases} \qquad \beta_{n,k}(t) = \langle e_{n,k}, \mathbb{1}_{[0,t]}\rangle,$$

and $\xi_{n,k} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$. Define $\Delta B_t^n = \sum_{k=0}^{2^n-1} \xi_{n,k}\beta_{n,k}(t)$ and $B_t^N = \sum_{n=0}^{N} \Delta B_t^n$.

1. Show that $\{e_{n,k}\}$ is orthonormal, i.e.,

$$\int_0^1 e_{n,k}(x)e_{n',k'}(x)\,dx = \mathbb{1}_{n=n'}\mathbb{1}_{k=k'}.$$

2. Show that

$$\sup_{t\in[0,1]} |\Delta B_t^n| \le 2^{-n/2} \cdot \max_{0\le k\le 2^n-1} |\xi_{n,k}|.$$

Hint: note that for fixed n, $e_{n,k}$ has disjoint support for different k.

3. Use $\mathsf{P}(|\mathcal{N}(0,1)| \ge a) \le e^{-a^2/2}$ and Borel–Cantelli Lemma to show that with probability one, there is a random constant $n_0 = n_0(\omega)$ such that

$$|\xi_{n,k}| \le n, \quad \forall 0 \le k \le 2^n - 1,\ n \ge n_0.$$

4. Conclude that with probability 1, $\{B_t^N(\omega), t \in [0,1]\}_{N\ge 1}$ is Cauchy in $\mathcal{C}[0,1]$, that is,

$$\lim_{N,N'\to\infty} \sup_{t\in[0,1]} |B_t^N(\omega) - B_t^{N'}(\omega)| = 0, \quad \text{a.e. } \omega.$$