HW13

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Exercise 1 (Le Gall, Ex 9.16) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function, and assume that f is a difference of convex functions. Let X be a continuous semi-martingale and consider the continuous semi-martingale $Y_t = f(X_t)$.

1. Recall that for any continuous semi-martingale Z,

$$L_t^a(Z) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{a \le X_s \le a+\varepsilon\}} \, d\langle Z \rangle_s, \quad L_t^{a-}(Z) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{a-\varepsilon \le X_s \le a\}} \, d\langle Z \rangle_s.$$

Show that for every $a \in \mathbb{R}$, $L_t^a(Y) = f'_+(a) L_t^a(X)$, $L_t^{a-} = f'_-(a) L_t^{a-}(X)$.

Proof. By linearity and the condition f is increasing as a difference of convex functions, we have there exists a process $(V_t)_{t\geq 0}$ of finite variation such that for $t \geq 0$: $Y_t = f(X_t) = f(X_0) + \int_0^t f'_+(X_s)\,dX_s + V_t$, which means $d\langle Y\rangle_s = (f'_+(X_s))^2 d\langle X\rangle_s$. Then by approximation formula, $L_t^a(Y) = L_t^a(f(X)) = \lim\limits_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t 1_{\{a\le f(X_s)\le a+\varepsilon\}}\, d\langle f(X)\rangle_s$

$= \lim\limits_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t 1_{\{a\le f(X_s)\le a+\varepsilon\}} (f'_+(X_s))^2 d\langle X\rangle_s = \lim\limits_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\mathbb{R}} 1_{\{a\le f(b)\le a+\varepsilon\}} (f'_+(b))^2 L_t^b(X)\,db$

$= \lim\limits_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\mathbb{R}} 1_{\{a\le f(b+)\le a+\varepsilon\}}\cdot f'_+(b)\, L_t^b(X)\, df(b+) = f'_+(a) L_t^a(X)$. Similarly, there exists a F.V. process $(\tilde{V})_{t\geq 0}$ s.t. $Y_t = f(X_0) + \int_0^t f'_-(X_s)\,dX_s + \tilde{V}_t$, which means that $d\langle Y\rangle_s = (f'_-(X_s))^2 d\langle X\rangle_s$. By approximation formula, $L_t^{a-}(Y) = L_t^{a-}(Y) = \lim\limits_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t 1_{\{a-\varepsilon\le f(X_s)\le \varepsilon\}}\cdot (f'_-(X_s))^2 d\langle X\rangle_s = \lim\limits_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\mathbb{R}} 1_{\{a-\varepsilon\le f(b)\le\varepsilon\}}\cdot (f'_-(b))^2 L_t^b(X)\,db = \lim\limits_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\mathbb{R}} 1_{\{a-\varepsilon\le f(b-)\le\varepsilon\}}\cdot f'_-(b)\cdot L_t^b(X)\, df(b-) = f'_-(a-) L_t^{a-}(X) = f'_-(a) L_t^{a-}(X)$.

2. Show that if $X = B$ is a Brownian motion, $L_t^a(f(B))$ is continuous if and only if f is continuously differentiable.

Proof. Since $L^a(f(B))$ is right-continuous in $a \in \mathbb{R}$, then

$L_t^a(f(B))$ is continuous $\iff L_t^a(f(B)) = L_t^{a-}(f(B))$, $a \in \mathbb{R}$.

$\overset{\text{(by 1)}}{\iff} f'_+(a) L_t^a(B) = f'_-(a) L_t^{a-}(B)$, $a \in \mathbb{R}$. And $f'_+(\cdot)$ is continuous.

$\overset{(B(\cdot)\text{ is a c.l.m.})}{\iff} f'_+(a) = f'_-(a)$, $a \in \mathbb{R}$. And $f'_+(\cdot)$ is continuous.

$\iff f \in C^1(\mathbb{R})$.

Exercise 2 (Le Gall, Ex 9.25) Let $\rho : [0,\infty) \to [0,\infty)$ be a non-decreasing function such that the improper integral $\int_0^1 \frac{du}{\rho(u)}$ diverges. Consider the SDE

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt$$

where

$$|\sigma(x) - \sigma(y)|^2 \le \rho(|x-y|), \quad |b(x) - b(y)| \le |x-y|.$$

1. Let Y be a continuous semi-martingale such that for every $t > 0$,

$$\int_0^t \frac{d\langle Y\rangle_s}{\rho(|Y_s|)} < \infty, \text{a.s.}$$

Prove that $L_t^0(Y) = 0$ for every $t > 0$, a.s.

Proof. For each fixed $t>0$, $\int_0^1 \frac{L_t^a(Y)}{\rho(|a|)} da \leqslant \int_{\mathbb{R}} \frac{L_t^a(Y)}{\rho(|a|)} da = \int_0^t \frac{d\langle Y\rangle_s}{\rho(|Y_s|)} < \infty$ a.s. If there exists some $t_0 > 0$ s.t. $L_{t_0}^0(Y) > 0$ on some measurable set A with $P(A) > 0$, then for $\omega \in A$, $\exists\, \varepsilon(\omega) > 0$ s.t. $L_{t_0}^a(Y)(\omega) \geqslant \frac{1}{2} L_{t_0}^0(Y)(\omega) > 0$ for $a \in [0, \varepsilon(\omega)]$, which implies that $\int_0^1 \frac{L_{t_0}^a(Y)}{\rho(|a|)} da \geqslant \frac{1}{2} L_{t_0}^0(Y) \int_0^1 \frac{da}{\rho(|a|)} = +\infty$ on A, which contradicts the result above.

2. Let X, X' be two solutions of the SDE on the same probability space with the same driven Brownian motion. Use Generalized Itô's formula to show that

$$|X_t - X_t'| = |X_0 - X_0'| + \int_0^t (\sigma(X_s) - \sigma(X_s'))\, \mathrm{sgn}(X_s - X_s')\, dB_s + \int_0^t (b(X_s) - b(X_s'))\, \mathrm{sgn}(X_s - X_s')\, ds.$$

Proof. By Generalized Itô's formula, for $t \geqslant 0$, $|X_t - X_t'| = |X_0 - X_0'| + \int_0^t \mathrm{sgn}(X_s - X_s')\, dX_s + L_t^0(X - X') = |X_0 - X_0'| + \int_0^t \mathrm{sgn}(X_s - X_s')(\sigma(X_s) - \sigma(X_s'))\, dB_s + \int_0^t \mathrm{sgn}(X_s - X_s')\cdot(b(X_s) - b(X_s'))\, ds + L_t^0(X - X')$. Thus it suffices to show $L_t^0(X - X') = 0$ for $t \geqslant 0$. Actually $d\langle X - X'\rangle_s = d\langle \int_0^\cdot (\sigma(X_r) - \sigma(X_r'))\, dB_r\rangle_s = (\sigma(X_s) - \sigma(X_s'))^2 ds$, then we have that for $t \geqslant 0$, $\int_0^t \frac{d\langle X - X'\rangle_s}{\rho(|X_s - X_s'|)} = \int_0^t \frac{(\sigma(X_s) - \sigma(X_s'))^2}{\rho(|X_s - X_s'|)} ds \leqslant \int_0^t 1\, ds = t < \infty$ a.s., which implies via 1.) that $L_t^0(X - X') = 0$ for $t \geqslant 0$. The proof is done.

3. Show that pathwise uniqueness holds for the SDE.

Proof. It is a direct consequence of Ex 1 of HW 10. OR we could apply results in 2.) above by making some localization procedures then taking expectation, we obtain $\mathbb{E}|X_t - X_t'| \leqslant \int_0^t \mathbb{E}|b(X_s) - b(X_s')|\, ds \leqslant \int_0^t \mathbb{E}|X_s - X_s'|\, ds$. Then the proof is done by Gronwall's inequality.

And there is a suggested solution to these two exercises by Te-Chun Wang from National Chiao Tung University.

9.1 Exercise 9.16

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a monotone increasing function, and assume that f is a difference of convex functions. Let X be a semimartingale and consider the semimartingale $Y_t = f(X_t)$. Prove that, for every $a \in \mathbb{R}$,

$$L_t^a(Y) = f'_+(a)L_t^a(X) \text{ and } L_t^{a-}(Y) = f'_-(a)L_t^{a-}(X).$$

In particular, if X is a Brownian motion, the local times of $f(X)$ are continuous in the space variable if and only if f is continuously differentiable.

Remark.
Note that $(L^a(X), a \in \mathbb{R})$ is the càdlàg modification of local time of X. The formula

$$L_t^a(Y) = f'_+(a)L_t^a(X)$$

doesn't hold for all increasing function $f = \varphi_1 - \varphi_2$, where φ_i is a convex function on $\mathbb{R}$. For example, if $\varphi_1(x) = 2e^x$ and $\varphi_2(x) = e^x$, and if X is a continuous semimartingale such that $\boldsymbol{P}(L_t^a(X) \neq 0) > 0$ for some $a < 0$ and $t > 0$, then $f(x) = e^x$ and so

$$L_t^a(Y) = L_t^a(f(X)) = 0 \neq e^a L_t^a(X) = f'(a)L_t^a(X)$$

on $\{L_t^a(X) \neq 0\}$.
To avoid this problem, we restatement Exercise 9.16 as following: Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a strictly increasing function such that $f = \varphi_1 - \varphi_2$, where φ_i is a convex function on $\mathbb{R}$. Let X be a semimartingale and consider the semimartingale $Y_t = f(X_t)$. Prove that, a.s.

$$L_t^{f(a)}(Y) = f'_+(a)L_t^a(X) \text{ and } L_t^{f(a)-}(Y) = f'_-(a)L_t^{a-}(X) \quad \forall a \in \mathbb{R}, t \geq 0$$

In particular, if X is a Brownian motion and $(u, v) \subseteq R(f) := \{a \in \mathbb{R} \mid f(a)\}$, we have, a.s. $a \in (u, v) \mapsto L^a(Y)$ is continuous if and only if $a \in (u, v) \mapsto f(a)$ is continuously differentiable.

Proof.

1. Since $f = \varphi_1 - \varphi_2$, we see that f is continuous and f'_+ is right continuous. We show that, a.s.

$$L_t^{f(a)}(Y) = f'_+(a)L_t^a(X) \quad \forall t > 0, a \in \mathbb{R}.$$

To show this, it suffices to show that $\boldsymbol{P}(L_t^{f(a)}(Y) = f'_+(a)L_t^a(X)) = 1$ for all $t \geq 0$ and $a \in \mathbb{R}$. Indeed, since $a \in \mathbb{R} \mapsto f'_+(a)L_t^a(X)$ is right continuous for $t \geq 0$ and

$$E_a := \{L_t^{f(a)}(Y) = f'_+(a)L_t^a(X) \quad \forall t \geq 0\} = \bigcap_{s \in \mathbb{Q}_+} E_{a,s} \quad \forall a \in \mathbb{R},$$

where

$$E_{a,s} := \{L_s^{f(a)}(Y) = f'_+(a)L_s^a(X)\} \quad \forall a \in \mathbb{R}, s > 0,$$

we see that

$$\boldsymbol{P}(L_t^{f(a)}(Y) = f'_+(a)L_t^a(X) \quad \forall a \in \mathbb{R}, t \geq 0) = \boldsymbol{P}(\bigcap_{q \in \mathbb{Q}} E_q) = 1.$$

Fix $a \in \mathbb{R}$ and $t > 0$. Now, we show that $\boldsymbol{P}(L_t^{f(a)}(Y) = f'_+(a)L_t^a(X)) = 1$. By generalized Itô formula, we see that

$$d\langle Y, Y\rangle_s = f'_-(X_s)^2 d\langle X, X\rangle_s.$$

By Proposition 9.9 and Corollary 9.7, we have, a.s.

$$\begin{aligned}
L_t^{f(a)}(Y) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t 1_{\{f(a)\le f(X_s)\le f(a)+\epsilon\}} f'_-(X_s)^2 d\langle X, X\rangle_s \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathbb{R}} 1_{\{f(a)\le f(b)\le f(a)+\epsilon\}} f'_-(b)^2 L_t^b(X)db \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathbb{R}} 1_{\{f(a)\le f(b)\le f(a)+\epsilon\}} f'_+(b)^2 L_t^b(X)db.
\end{aligned}$$

We show that, a.s.

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathbb{R}} 1_{\{f(a)\le f(b)\le f(a)+\epsilon\}} f'_+(b)^2 L_t^b(X)db = f'_+(a)L_t^a(X).$$

Fix w. Given $\eta > 0$. Choose $h > 0$ such that

$$|f'_+(a)L_t^a(X) - f'_+(b)L_t^b(X)| < \eta$$

whenever $a \le b < a+h$. Note that f is a continuous strictly increasing function. For $\epsilon > 0$, define

$$a_\epsilon := \inf\{b \in \mathbb{R} \mid f(b) = f(a)+\epsilon\}.$$

Choose $j > 0$ such that $a < a_\epsilon < a+h$ for every $0 < \epsilon < j$. Let $0 < \epsilon < j$. Then $-\infty < a < a_\epsilon < \infty$, $f(a_\epsilon) = f(a)+\epsilon$,

$$|f'_+(a)L_t^a(X) - f'_+(b)L_t^b(X)| < \eta \quad \forall b \in [a, a_\epsilon],$$
$$\{b \in \mathbb{R} \mid f(a) \le f(b) \le f(a)+\epsilon\} = [a, a_\epsilon],$$

and so

$$\frac{1}{\epsilon}\int 1_{\{f(a)\le f(b)\le f(a)+\epsilon\}} f'_+(b)db = \frac{1}{\epsilon}\int_a^{a_\epsilon} f'_+(b)db = \frac{f(a_\epsilon)-f(a)}{\epsilon} = 1.$$

Thus,

$$\begin{aligned}
&|\frac{1}{\epsilon}\int_{\mathbb{R}} 1_{\{a\le f(b)\le a+\epsilon\}} f'_+(b)^2 L_t^b(X)db - f'_+(a)L_t^a(X)| \\
&= |\frac{1}{\epsilon}\int_a^{a_\epsilon} f'_+(b)^2 L_t^b(X)db - \frac{1}{\epsilon}\int_a^{a_\epsilon} f'_+(b) f'_+(a)L_t^a(X)db| \\
&\le \frac{1}{\epsilon}\int_a^{a_\epsilon} f'_+(b)|f'_+(b)L_t^b(X) - f'_+(a)L_t^a(X)|db \\
&< \eta\frac{1}{\epsilon}\int_a^{a_\epsilon} f'_+(b)db = \eta\frac{1}{\epsilon}(f(a_\epsilon)-f(a)) = \eta\frac{1}{\epsilon}\epsilon = \eta.
\end{aligned}$$

Therefore, we have, a.s.

$$L_t^{f(a)}(Y) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathbb{R}} 1_{\{f(a)\le f(b)\le f(a)+\epsilon\}} f'_+(b)^2 L_t^b(X)db = f'_+(a)L_t^a(X).$$

2. We show that, a.s.

$$L_t^{f(a)-}(Y) = f'_-(a)L_t^{a-}(X) \quad \forall t > 0, a \in \mathbb{R}.$$

To show this, it suffices to show that $\lim_{b\uparrow a} f'_+(b) = f'_-(a)$ for every $a \in \mathbb{R}$. Indeed, if $w \in E$, where $E = \{L_t^{f(a)}(Y) = f'_+(a)L_t^a(X) \quad \forall a \in \mathbb{R}, t \ge 0\}$, then

$$L_t^{f(a)-}(Y) = \lim_{b\uparrow a} L_t^{f(b)}(Y) = \lim_{b\uparrow a} f'_+(b)L_t^b(X) = f'_-(a)L_t^{a-}(X) \quad \forall a \in \mathbb{R}, t \ge 0.$$

Fix $a \in \mathbb{R}$. Now, we show that $\lim_{b\uparrow a} f'_+(b) = f'_-(a)$. Since $f = \varphi_1 - \varphi_2$, it suffices to show that $\lim_{b\uparrow a} \varphi'_{i,+}(b) = \varphi'_{i,-}(a)$ for $i = 1, 2$. We denote φ_i as φ. It's clear that

$$\varphi'_+(b) \leq \varphi'_-(a) \quad \forall b < a.$$

Given $\eta > 0$. There exists $c < a$ such that

$$\varphi'_-(a) - \eta \leq \frac{\varphi(a) - \varphi(c)}{a - c}.$$

By continuity, there exists $c < d < a$ such that

$$\frac{\varphi(a) - \varphi(c)}{a - c} - \eta \leq \frac{\varphi(d) - \varphi(c)}{d - c}$$

and so

$$\varphi'_-(a) - 2\eta \leq \frac{\varphi(d) - \varphi(c)}{d - c} \leq \varphi'_+(b) \quad \forall d < b < a.$$

Thus, we get

$$\varphi'_-(a) - 2\eta \leq \varphi'_+(b) \leq \varphi'_-(a) \quad \forall d < b < a$$

and, hence, $\lim_{b\uparrow a} f'_+(b) = f'_-(a)$.

9.10 Exercise 9.25 (Another look at the Yamada–Watanabe criterion)

Let ρ be a nondecreasing function from $[0, \infty)$ into $[0, \infty)$ such that, for every $\epsilon > 0$,

$$\int_0^\epsilon \frac{du}{\rho(u)} = \infty.$$

Consider then the one-dimensional stochastic differential equation

$$E(\sigma, b): \qquad dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

where one assumes that the functions σ and b satisfy the conditions

$$(\sigma(x) - \sigma(y))^2 \leq \rho(|x - y|), \quad |b(x) - b(y)| \leq K|x - y|,$$

for every $x, y \in \mathbb{R}$, with a constant $K < \infty$. Our goal is use local times to give a short proof of pathwise uniqueness for $E(\sigma, b)$ (this is slightly stronger than the result of Exercise 8.14).

1. Let Y be a continuous semimartingale such that, for every $t > 0$,
$$\int_0^t \frac{d\langle Y, Y\rangle_s}{\rho(|Y_s|)} < \infty \quad (a.s.).$$
Prove that $L_t^0(Y) = 0$ for every $t \geq 0$ (a.s.).
2. Let X and X_0 be two solutions of $E(\sigma, b)$ on the same filtered probability space and with the same Brownian motion B. By applying question 1. to $Y = X - X'$, prove that $L_t^0(X - X')$ for every $t \geq 0$ (a.s.) and therefore,
$$|X_t - X'_t| = |X_0 - X'_0| + \int_0^t (\sigma(X_s) - \sigma(X'_s))sgn(X_s - X'_s)dB_s + \int_0^t (b(X_s) - b(X'_s))sgn(X_s - X'_s)ds.$$
3. Using Gromwall's lemma, prove that if $X_0 = X'_0$, then $X_t = X'_t$ for every $t \geq 0$ (a.s.).

Proof.

1. Since $L_t^a(Y) \overset{a\downarrow 0}{\to} L_t^0(Y) \quad \forall t \geq 0$ (a.s.), there exists $C = C(w) > 0$ and $\epsilon = \epsilon(w) > 0$ such that
$$L_t^a(Y) \geq CL_t^0(Y) \quad \forall 0 < a < \epsilon \quad \forall t \geq 0 \quad (a.s.).$$
By Density of occupation time formula (Corollary 9.7), we have
$$\infty > \int_0^t \frac{d\langle Y, Y\rangle_s}{\rho(|Y_s|)} = \int_{\mathbb{R}} \frac{1}{\rho(|a|)} L_t^a(Y) da \geq CL_t^0(Y) \int_0^\epsilon \frac{1}{\rho(a)} da \quad \forall t \geq 0 \quad (a.s.).$$
Since $\int_0^\epsilon \frac{du}{\rho(u)} = \infty$ for all $\epsilon > 0$, we get $L_t^0(Y) = 0$ for all $t \geq 0$ (a.s.).

2. Set $Y = X - X'$. Then
$$Y_t = X_0 - X_0' + \int_0^t (\sigma(X_s) - \sigma(X_s'))dB_s + \int_0^t (b(X_s) - b(X_s'))ds$$
and so
$$d\langle Y, Y\rangle_t = (\sigma(X_t) - \sigma(X_t'))^2 dt.$$
Thus,
$$\int_0^t \frac{d\langle Y, Y\rangle_s}{\rho(|Y_s|)} = \int_0^t \frac{(\sigma(X_s) - \sigma(X_s'))^2}{\rho(|X_s - X_s'|)} ds \leq \int_0^t \frac{\rho(|X_s - X_s'|)}{\rho(|X_s - X_s'|)} ds = t < \infty \quad \forall t \geq 0 \quad (a.s.).$$
By question 1., we get $L_t^0(X - X') = 0$ for every $t \geq 0$ (a.s.). By Tanaka's formula, we have
$$|X_t - X_t'| = |X_0 - X_0'| + \int_0^t (\sigma(X_s) - \sigma(X_s'))sgn(X_s - X_s')dB_s + \int_0^t (b(X_s) - b(X_s'))sgn(X_s - X_s')ds$$
for every $t \geq 0$ (a.s.).

3. By continuity, it suffices to show that $X_t = X_t'$ (a.s.) for every $t \geq 0$. Fix $t_0 > 0$ and choose $L > t_0$. Define
$$T_M = \inf\{s \geq 0 \mid |X_s| \geq M \text{ or } |X_s'| \geq M\} \quad \forall M > 0.$$
Fix $M > 0$. Since
$$\begin{aligned}&\boldsymbol{E}[\langle \int_0^\cdot (\sigma(X_s) - \sigma(X_s'))sgn(X_s - X_s')1_{[0,T_M]}dB_s, \int_0^\cdot (\sigma(X_s) - \sigma(X_s'))sgn(X_s - X_s')1_{[0,T_M]}dB_s\rangle_t] \\ &= \boldsymbol{E}[\int_0^t (\sigma(X_s) - \sigma(X_s'))^2 1_{[0,T_M]}ds] \leq \boldsymbol{E}[\int_0^t \rho(|X_s - X_s'|)1_{[0,T_M]}ds] \leq \rho(2M)t < \infty \quad \forall t > 0,\end{aligned}$$
and
$$g(t) = \boldsymbol{E}[|X_t - X_t'|1_{[0,T_M]}(t)] = \boldsymbol{E}[\int_0^t (b(X_s) - b(X_s'))sgn(X_s - X_s')1_{[0,T_M]}ds] \leq 2K \int_0^t g(s)ds$$
for every $t \in [0, L]$. By Gromwall's lemma, we get $g(t) = 0$ in $[0, L]$ and so $\boldsymbol{E}[|X_{t_0\wedge T_M} - X'_{t_0\wedge T_M}|] = 0$. By letting $M \uparrow \infty$, we have $\boldsymbol{E}[|X_{t_0} - X'_{t_0}|] = 0$ and so $X_{t_0} = X'_{t_0}$.

□