HW13

May 22, 2024

Exercise 1 (Le Gall, Ex 9.16) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function, and assume that f is a difference of convex functions. Let X be a continuous semi-martingale and consider the continuous semi-martingale $Y_t = f(X_t)$.

1. Recall that for any continuous semi-martingale Z,

$$L_t^a(Z) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{a \le X_s \le a+\varepsilon\}} \, d\langle Z \rangle_s, \quad L_t^{a-}(Z) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{a-\varepsilon \le X_s \le a\}} \, d\langle Z \rangle_s.$$

   Show that for every $a \in \mathbb{R}$, $L_t^a(Y) = f'_+(a) L_t^a(X)$, $L_t^{a-} = f'_-(a) L_t^{a-}(X)$.

2. Show that if $X = B$ is a Brownian motion, $L_t^a\big(f(B)\big)$ is continuous if and only if f is continuously differentiable.

Exercise 2 (Le Gall, Ex 9.25) Let $\rho : [0, \infty) \to [0, \infty)$ be a non-decreasing function such that the improper integral $\displaystyle\int_0^1 \frac{du}{\rho(u)}$ diverges. Consider the SDE

$$dX_t = \sigma(X_t) \, dB_t + b(X_t) \, dt$$

where

$$|\sigma(x) - \sigma(y)|^2 \le \rho(|x - y|), \quad |b(x) - b(y)| \le |x - y|.$$

1. Let Y be a continuous semi-martingale such that for every $t > 0$,

$$\int_0^t \frac{d\langle Y \rangle_s}{\rho(|Y_s|)} < \infty, \text{a.s.}$$

   Prove that $L_t^0(Y) = 0$ for every $t > 0$, a.s.

2. Let X, X' be two solutions of the SDE on the same probability space with the same driven Brownian motion. Use Generalized Itô's formula to show that

$$|X_t - X'_t| = |X_0 - X'_0| + \int_0^t \big(\sigma(X_s) - \sigma(X'_s)\big) \operatorname{sgn}(X_s - X'_s) \, dB_s + \int_0^t \big(b(X_s) - b(X'_s)\big) \operatorname{sgn}(X_s - X'_s) \, ds.$$

3. Show that pathwise uniqueness holds for the SDE.