HW12

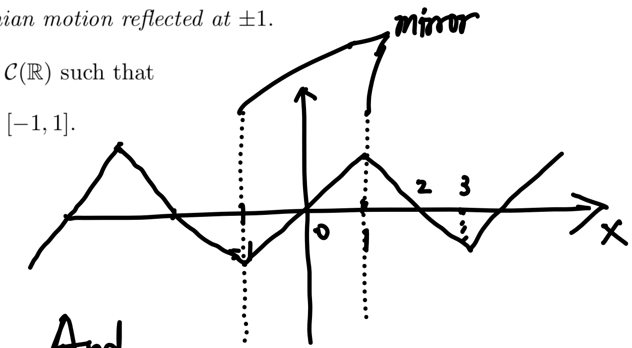
May 15, 2024

Exercise 1 Let $\phi : \mathbb{R} \to [-1,1]$ be defined as

$$\phi(x) = \begin{cases} x - 4k, & x \in (4k-1, 4k+1], \\ 4k+2-x, & x \in (4k+1, 4k+3]. \end{cases}$$

That is, on the real line, if two mirrors are placed at ± 1 and $x \in [-1,1]$, then $\phi^{-1}(\{x\})$ is the location of the images of x.

Let B_t be a Brownian motion and $X_t = \phi(B_t)$ be the *Brownian motion reflected at* ± 1.



- Let $f \in \mathcal{C}^2(-1,1) \cap \mathcal{C}^1[-1,1]$. Show that there exists $f_\phi \in \mathcal{C}(\mathbb{R})$ such that

$$\mathsf{E}^x f(X_t) = \mathsf{E}^x f_\phi(B_t), \quad \forall x \in [-1,1].$$

Proof. By definition $\phi(\cdot) \in C(\mathbb{R} : [-1,1])$, and $f \in C^2(-1,1) \cap C^1[-1,1] \subseteq C([-1,1])$, thus $f \circ \phi \in C(\mathbb{R})$. And for $x \in [-1,1]$, $\mathbb{E}^x f(X_t) = \mathbb{E}^x f \circ \phi(B_t)$. Set $f_\phi = f \circ \phi$, the proof is done.

- Show that $f'(\pm 1) = 0$ if and only if $f_\phi \in \mathcal{C}^2(\mathbb{R})$.

  Remark: The first two parts in fact give that X_t has generator

$$\mathcal{L}^X = \frac{1}{2}\partial_{xx}, \quad \mathcal{D}(\mathcal{L}^X) = \{g \in \mathcal{C}[-1,1] : g', g'' \in \mathcal{C}[-1,1], g'(\pm 1) = 0\}.$$

Proof. Firstly suppose $f'(\pm1)=0$. By periodicity of ϕ, to prove $f_\phi \in C^2(\mathbb{R})$, it suffices to show that $f_\phi''(x_0)$ exists for $x_0 \in [-1,3)$. For $x_0 \in (-1,1)$, $\lim_{x\to x_0} \frac{f_\phi(x)-f_\phi(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f\circ\phi(x) - f\circ\phi(x_0)}{x-x_0}$ $= \lim_{\phi(x)\to\phi(x_0)} \frac{f(\phi(x))-f(\phi(x_0))}{\phi(x)-\phi(x_0)} \cdot \lim_{x\to x_0}\frac{\phi(x)-\phi(x_0)}{x-x_0} = f'(\phi(x_0))\cdot\phi'(x_0) = f'(\phi(x_0))\cdot 1$ exists since $f \in C^1[-1,1]$. Similarly, for $x_0 \in (1,3)$, $\lim_{x\to x_0}\frac{f_\phi(x)-f_\phi(x_0)}{x-x_0} = f'(\phi(x_0))\cdot\phi'(x_0) = f'(\phi(x_0))\cdot(-1)$.

Then for $x_0 \in (-1,1]$, $f_\phi''(x_0) = \lim_{x\to x_0}\frac{f_\phi'(x)-f_\phi'(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{f'(\phi(x))-f'(\phi(x_0))}{x-x_0} = f''(\phi(x_0))(\phi'(x_0))^2 =$ $f''(\phi(x_0))$ exists since $f \in C^2(-1,1)$. Similarly we have for $x_0 \in (1,3)$, $f_\phi''(x_0)$ exists and continuous at x_0.

Now consider the case that $x_0 \in \{-1,1\}$. We claim $f_\phi'(1)=0$. Since

$|f_\phi'(1)| \leq \limsup_{x\to1}\left|\frac{f_\phi(x)-f_\phi(1)}{x-1}\right| = \limsup_{x\to1}\left|\frac{f\circ\phi(x)-f\circ\phi(1)}{x-1}\right| = \lim_{\phi(x)\to\phi(1)}\left|\frac{f(\phi(x))-f(\phi(1))}{\phi(x)-\phi(1)}\right| \cdot \limsup_{x\to1}\left|\frac{\phi(x)-\phi(1)}{x-1}\right|$

$\leq f'(\phi(1))\cdot\limsup_{x\to1}\frac{|\phi(x)-\phi(1)|}{|x-1|} = f'(1)\cdot 1 = 0$. Similarly we have $\lim_{x\to-1}\left|\frac{f_\phi(x)-f_\phi(-1)}{x+1}\right|$ $= f'(\phi(-1))\cdot 1 = 0$. Then $f_\phi''(1) = \lim_{x\to1}\frac{f_\phi'(x)-f_\phi'(1)}{x-1} = \lim_{x\to1}\frac{f'(\phi(x))\phi'(x)}{x-1} = \lim_{x\to1} f''(\phi(x))\phi'(x)^2$
(L'Hôpital)

(some arguments missing here. See below)

$\boxed{\text{exists}}$. Similarly we have $f''_\phi(-1) = \lim_{x\to -1} f''_\phi(x)$. Thus $f_\phi \in C^2(\mathbb{R})$.

(Note that some eqs above have used the fact that ϕ is symmetric w.r.t. $x = x_0$ in some neighborhood of x_0, where $x_0 \in \{-1, 1\}$.) Conversely, suppose $f_\phi = f\circ\phi \in C^2(\mathbb{R})$. Then $f''_\phi(1)$ exists, which implies that $f''_\phi(1) = \lim_{h\to 0} \frac{f_\phi(1+h) + f_\phi(1-h) - 2f_\phi(1)}{h^2}$ $= \lim_{h\to 0} \frac{2(f_\phi(1+h) - f_\phi(1))}{h^2} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{f_\phi(1+h) - f_\phi(1)}{h}$, which means $f'_\phi(1) = \lim_{h\to 0} h \cdot f''_\phi(1) = 0$. Then $0 = f'_\phi(1) = f'(\phi(1)) \cdot \limsup_{x\to 1} \frac{\phi(x) - \phi(1)}{x-1} = f'(1)$. Similarly we have $f'(-1) = 0$. The proof is done.

- Let u be a classical solution to the heat equation with Neumann boundary condition:

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\partial_{xx} u(t,x), & t > 0, x \in (-1,1), \\ \partial_x u(t, \pm 1) = 0, \\ u(0,x) = f(x), & x \in (-1,1), \end{cases}$$

where $f \in \mathcal{D}(\mathcal{L}^X)$. Show that we have the representation

$$u(t,x) = \mathsf{E}^x f(X_t).$$

Proof. For a fixed $t > 0$, define for $s \in [0,t]$ that $Y_s := u(t-s, X_s) = u(t-s, \phi(B_s))$. By condition $\partial_x u(t, \pm 1) = 0$, $t \geq 0$ (since u is a classical solution $\partial_x u$ could be continuously extended to $\{t=0\}$). By results above we know that the map $(s,x) \mapsto u(t-s, \phi(x))$ is in $C^{1,2}([0,+\infty), \mathbb{R})$. By Itô's formula, $Y_s = u(t-s, X_s) = u(t, X_0) + \int_0^s -\partial_t u(t-\tau, X_\tau)\,d\tau + \int_0^s \partial_x u(t-\tau, X_\tau)\phi'(B_\tau)\cdot 1_{\{\tau \neq 1+4k,\, 3+4k\}}\,dB_\tau +$ $\int_0^s \frac{1}{2}\partial_{xx} u(t-\tau, X_\tau)(\phi'(B_\tau))^2 1_{\{\tau\neq 1+4k,\, 3+4k\}}\,d\langle B\rangle_\tau = u(t, X_0) + \int_0^s \partial_x(u(t-\tau, \phi(B_\tau)))\,dB_\tau$, which means $(Y_s)_{s\in[0,t]}$ is a martingale under P^x, $x \in [-1,1]$. Then $\mathbb{E}^x f(X_t) = \mathbb{E}^x u(0, X_t) = \mathbb{E}^x Y_t = \mathbb{E}^x Y_0 =$ $\mathbb{E}^x u(t, X_0) = u(t,x)$. The proof is done.

- By separation of variables, solution to the PDE can be written as

$$u(t,x) = \sum_{n=1}^{\infty} c_n(t) e^{-\lambda_n t} e_n(t)$$

where $e_n(t)$ are some trigonometric functions. Use this to show that

$$\lim_{t\to\infty} u(t,x) = \frac{1}{2}\int_{-1}^{1} f(y)\,dy.$$

Proof. According to separation of variables, assume a priori $u(t,x) = A(t)\cdot B(x)$ for some function $A(\cdot)$ and $B(\cdot)$. Then $A'(t)B(x) = \frac{1}{2}A(t)B''(x)$, i.e. $2\frac{A'(t)}{A(t)} = \frac{B''(x)}{B(x)} = C$ for some $C \in \mathbb{R}$. Then we have $\begin{cases} A' = \frac{1}{2}CA, \\ B'' = CB, \end{cases}$ which implies $A(t) = C_0 e^{\frac{1}{2}Ct}$, $C_0 \in \mathbb{C}$; $B(x) = C_1 + C_2 x$ if $C = 0$ or $B(x) = C_1 \cos\sqrt{-\frac{1}{2}C}\, x + C_2 \sin\sqrt{-\frac{1}{2}C}\, x$ if $C \neq 0$, where $C_1, C_2 \in \mathbb{C}$. Since $u(0,x) = f(x)$ may not be a constant we take $C \neq 0$. By condition $\partial_x u(t, \pm 1) = 0$, (*) we have $C_2 \cos\sqrt{-\frac{1}{2}C} = C_1 \sin\sqrt{-\frac{1}{2}C}$, which implies that $\exists\, \lambda > 0$ s.t. $\sqrt{-\frac{1}{2}C} = n\lambda\pi$, $n \in \mathbb{Z}$. Thus $C = -2(n\lambda\pi)^2 < 0$ for $n \neq 0$ and $C = 0$ for $n = 0$.

Also since $\sin\sqrt{-\frac{1}{2}C} = \frac{C_2}{C_1}\cos\sqrt{-\frac{1}{2}C}$, we could write $B(x) = C_3\cos\sqrt{-\frac{1}{2}C}\,x$, $C_3 \in K$. By the fact that $\cos(-x) = \cos x$ and principle of superposition, we could write the solution $u(t,x) = \sum_{n=0}^{\infty} C_n e^{-\lambda_n t}\cos n\lambda\pi x$, where $\lambda \in \mathbb{R}$, $\lambda_n > 0$ for $n \geqslant 1$ and $\lambda_0 = 0$; and $C_n \in \mathbb{R}$ for $n \in \mathbb{N}$. Then obviously $\lim_{t\to+\infty} u(t,x) = C_0$. And $f(x) = u(0,x) = \sum_{n=0}^{\infty} C_n \cos n\lambda\pi x$. By (*) we know $(-1,1)$ is a periodic interval. By properties of Fourier series we have $C_0 = \frac{1}{2}\int_{-1}^{1} f(y)dy$. The proof is done.

- Show that starting from any initial condition $\mu \in \mathcal{M}[-1,1]$, we have the convergence in distribution

$$\mathcal{L}(X_t) = \mathsf{P}^{\mu}(X_t \in \cdot) \to \mathrm{Unif}[-1,1].$$

Proof. By definition of P^{μ}, we have for $t \geqslant 0$, $P^{\mu}(X_t \in \cdot) = \int_{-1}^{1} P^x(X_t \in \cdot)\mu(dx)$ (*). And by results above, we have $\mathbb{E}^x f(X_t) = u(t,x) \xrightarrow{t\to+\infty} \int_{-1}^{1}\frac{1}{2}f(y)dy$ (*) for all $f \in D(L^X)$. By density of $D(L^X)$ in $C_0([-1,1])$ under the uniform norm and D.C.T. we know that (*) holds for all $f \in C_0([-1,1])$. Thus $P^x(X_t \in \cdot) \xrightarrow{t\to+\infty} \mathrm{Unif}[-1,1]$ vaguely. By (*) we have for $a \in [-1,1]$, $P^{\mu}(X_t \in [-1,a]) = \int_{-1}^{1} P^x(X_t \in [-1,a])\mu(dx) \xrightarrow{t\to+\infty} a+1 \cdot \mu([-1,1]) = a+1$ $= \mathrm{Unif}([-a,1])$ by D.C.T. and the fact that X_t has continuous distribution. Thus $P^{\mu}(X_t \in \cdot) \longrightarrow \mathrm{Unif}[-1,1]$ vaguely.

Or we could delete it and restrict 'a' in a dense subset of $[-1,1]$, the same result holds.

Exercise 2 Recall that we have construct the Bessel-3 process

$$dX_t = dW_t + \frac{1}{X_t}dt$$

as *Brownian motion conditioned on never hitting 0* via the Doob's h-transform. The diffusion X has generator $\mathcal{L} = \frac{1}{2}\partial_{xx} + \frac{1}{x}\partial_x$.

- For $0 < \varepsilon < R$, let

$$u_{\varepsilon,R}(x) = \mathsf{P}^x(\tau_\varepsilon < \tau_R), \quad x \in (\varepsilon, R).$$

Find the exact form of $u_{\varepsilon,R\}}$ by solve the ODE $\mathcal{L}u = 0$ with boundary condition $u_{\varepsilon,R}(\varepsilon) = 1$, $u_{\varepsilon,R}(R) = 0$.

Proof. (Formally) $Lu_{\varepsilon,R} = \frac{1}{2}u''_{\varepsilon,R} + \frac{1}{x}u'_{\varepsilon,R} = 0 \Rightarrow \frac{1}{2}\frac{du'_{\varepsilon,R}}{dx} = -\frac{1}{x}u'_{\varepsilon,R} \Rightarrow \frac{1}{2}\frac{du'_{\varepsilon,R}}{u'_{\varepsilon,R}} = -\frac{1}{x}dx$

$\Rightarrow \frac{1}{2}\log u'_{\varepsilon,R} = -\log x + C \Rightarrow u'_{\varepsilon,R} = C_1\frac{1}{x^2} \Rightarrow u_{\varepsilon,R}(x) = C_2\frac{1}{x} + C_3$, $(C, C_1, C_2, C_3 \in \mathbb{R})$.

By $\begin{cases} u_{\varepsilon,R}(\varepsilon) = 1, \\ u_{\varepsilon,R}(R) = 0, \end{cases}$ we have $\begin{cases} C_2 = \frac{\varepsilon R}{R-\varepsilon}, \\ C_3 = \frac{-\varepsilon}{R-\varepsilon}, \end{cases}$ thus $u_{\varepsilon,R}(x) = \frac{\varepsilon R}{R-\varepsilon}\cdot\frac{1}{x} - \frac{\varepsilon}{R-\varepsilon}$, $x \in [-1,1]$.

- Find $\mathsf{P}^x(\tau_\varepsilon < \infty)$.

Proof. $P^x(\tau_\varepsilon < \infty) = P^x\left(\bigcup_{n=[\varepsilon]+1}^{\infty}\{\tau_\varepsilon < \tau_n\}\right) = \lim_{n\to+\infty} u_{\varepsilon,n}(x) = \lim_{n\to+\infty}\frac{\varepsilon n}{n-\varepsilon}\cdot\frac{1}{x} - \frac{\varepsilon}{n-\varepsilon} = \frac{\varepsilon}{x}$.

- Find a positive function $h(x)$ such that $\mathcal{L}h = 0$ and

$$\frac{h(x)}{h(y)} = \lim_{\varepsilon\to 0} \frac{\mathsf{P}^x(\tau_\varepsilon < \infty)}{\mathsf{P}^y(\tau_\varepsilon < \infty)}.$$

Proof. By results above we have for $x, y \in (-1,1)$, $\lim_{\varepsilon\to 0} \frac{P^x(\tau_\varepsilon<\infty)}{P^y(\tau_\varepsilon<\infty)} = \frac{\frac{\varepsilon}{x}}{\frac{\varepsilon}{y}} = \frac{y}{x}$, which implies that we could set $h(x) = \frac{1}{x}$. And obviously $h(\cdot)$ satisfies $Lh = 0$.

- Compute the generator

$$(\mathcal{L}^h f)(x) = \frac{1}{h(x)}(\mathcal{L}hf)(x).$$

This corresponds to the Bessel-3 process *conditioned on hitting* 0 *in finite time*. What is the conditioned process?

Proof. By results above and Doob's h-transform, we have

$$L^h f(x) = \frac{1}{h(x)} L(hf)(x) = \frac{1}{2h(x)}\cdot(hf)''(x) + \frac{1}{xh(x)}(hf)'(x)$$

$$= \frac{1}{2h(x)}(h'f + hf')'(x) + (h'f + hf')(x)$$

$$= \frac{1}{2h(x)}(h''f + h'f' + h'f' + hf'')(x) + (h'f + hf')(x)$$

$$= \left(\frac{h''}{2h}f + \frac{h'}{h}f' + \frac{1}{2}f''\right)(x) + (h'f + hf')(x)$$

$$= \frac{1}{2}f'' + \left(\frac{h'}{h} + h\right)f' + \left(\frac{h''}{2h} + h'\right)f\ (x)$$

$$= \frac{1}{2}f''(x) + 0 + 0$$

$$= \frac{1}{2}f''(x),$$

which means $L^h = \frac{1}{2}\partial_{xx}$ and the conditioned process is a standard one dimensional Brownian motion.

Correction to the proof of "$f'(\pm 1) = 0 \Rightarrow f_\phi \in C^2(\mathbb{R})$".

I forget to show $f_\phi''(\cdot)$ is continuous at $x_0 \in \{-1,1\}$ and $\lim_{x\to\pm 1} f_\phi''(x)$ exists. Actually since $f_\phi'(x) = f'(\phi(x))\cdot\phi'(x)$ for x in the neighborhood of x_0, and $\phi'(\cdot)$ is bounded, $f \in C^1[-1,1]$, thus $\lim_{x\to x_0} f'(\phi(x))\phi'(x) = 0 = f_\phi'(x_0)$ for $x_0 \in \{-1,1\}$. The existence of $\lim_{x\to 1} f''(\phi(x))$ may need the condition $f \in C^2[-1,1]$, i.e. f'' is uniformly continuous in $(-1,1)$.