HW12

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Exercise 1 Let $\phi : \mathbb{R} \to [-1, 1]$ be defined as

$$\phi(x) = \begin{cases} x - 4k, & x \in (4k-1, 4k+1], \\ 4k + 2 - x, & x \in (4k+1, 4k+3]. \end{cases}$$

That is, on the real line, if two mirrors are placed at ± 1 and $x \in [-1, 1]$, then $\phi^{-1}(\{x\})$ is the location of the images of x.

Let B_t be a Brownian motion and $X_t = \phi(B_t)$ be the *Brownian motion reflected at* ± 1.

- Let $f \in \mathcal{C}^2(-1, 1) \cap \mathcal{C}^1[-1, 1]$. Show that there exists $f_\phi \in \mathcal{C}(\mathbb{R})$ such that

$$\mathsf{E}^x f(X_t) = \mathsf{E}^x f_\phi(B_t), \quad \forall x \in [-1, 1].$$

- Show that $f'(\pm 1) = 0$ if and only if $f_\phi \in \mathcal{C}^2(\mathbb{R})$.

  Remark: The first two parts in fact give that X_t has generator

$$\mathcal{L}^X = \frac{1}{2}\partial_{xx}, \quad \mathcal{D}(\mathcal{L}^X) = \{g \in \mathcal{C}[-1, 1] : g', g'' \in \mathcal{C}[-1, 1], g'(\pm 1) = 0\}.$$

- Let u be a classical solution to the heat equation with Neumann boundary condition:

$$\begin{cases} \partial_t u(t, x) = \frac{1}{2}\partial_{xx} u(t, x), & t > 0, x \in (-1, 1), \\ \partial_x u(t, \pm 1) = 0, & \\ u(0, x) = f(x), & x \in (-1, 1), \end{cases}$$

  where $f \in \mathcal{D}(\mathcal{L}^X)$. Show that we have the representation

$$u(t, x) = \mathsf{E}^x f(X_t).$$

- By separation of variables, solution to the PDE can be written as

$$u(t, x) = \sum_{n=1}^{\infty} c_n(t) e^{-\lambda_n t} e_n(t)$$

  where $e_n(t)$ are some trigonometric functions. Use this to show that

$$\lim_{t \to \infty} u(t, x) = \frac{1}{2} \int_{-1}^{1} f(y)\, dy.$$

- Show that starting from any initial condition $\mu \in \mathcal{M}[-1,1]$, we have the convergence in distribution
$$\mathcal{L}(X_t) = \mathsf{P}^\mu(X_t \in \cdot) \to \mathrm{Unif}[-1,1].$$

Exercise 2 Recall that we have construct the Bessel-3 process

$$dX_t = dW_t + \frac{1}{X_t}\,dt$$

as *Brownian motion conditioned on never hitting 0* via the Doob's h-transform. The diffusion X has generator $\mathcal{L} = \frac{1}{2}\partial_{xx} + \frac{1}{x}\partial_x$.

- For $0 < \varepsilon < R$, let
$$u_{\varepsilon,R}(x) = \mathsf{P}^x(\tau_\varepsilon < \tau_R), \quad x \in (\varepsilon, R).$$
Find the exact form of $u_{\varepsilon,R\}}$ by solve the ODE $\mathcal{L}u = 0$ with boundary condition $u_{\varepsilon,R}(\varepsilon) = 1$, $u_{\varepsilon,R}(R) = 0$.

- Find $\mathsf{P}^x(\tau_\varepsilon < \infty)$.

- Find a positive function $h(x)$ such that $\mathcal{L}h = 0$ and
$$\frac{h(x)}{h(y)} = \lim_{\varepsilon\to 0} \frac{\mathsf{P}^x(\tau_\varepsilon < \infty)}{\mathsf{P}^y(\tau_\varepsilon < \infty)}.$$

- Compute the generator
$$(\mathcal{L}^h f)(x) = \frac{1}{h(x)}(\mathcal{L}hf)(x).$$
This corresponds to the Bessel-3 process *conditioned on hitting* 0 *in finite time.* What is the conditioned process?