HW11

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Exercise 1 Let $(M_t)_{t\geq 0}$ be a c.l.m. on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ with

$$\langle M\rangle_t = \int_0^t a(s)\,ds$$

for some progressively measurable process $a \geq 0$. Show that there exists an extended probability space $(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathsf{P}})$ and a standard Brownian motion B on it such that

$$M_t = \int_0^t \sqrt{a(s)}\,dB_s.$$

(For a proof of the multi-dimensional version, see Theorem 3.4.2 in GTM113.)

Hint: consider

$$B_t := \int_0^t \mathbb{1}_{\{a(s)>0\}}\frac{1}{\sqrt{a(s)}}\,dM_s + \int_0^t \mathbb{1}_{\{a(s)=0\}}\,dW_s$$

where W is a standard Brownian motion independent of everything else.

Proof. Let $(\hat\Omega, \hat{\mathcal{F}}, \hat{P})$ be a complete probability space on which a standard Brownian motion $(W_t)_{t\geq 0}$ is defined. Let $\hat{\mathcal{F}}_t := \sigma(W_s, 0\leq s\leq t)$, $t\geq 0$. Let $(\tilde\Omega, \tilde{\mathcal{G}}, \tilde{Q})$ be the product measure space s.t. $\tilde\Omega := \Omega\times\hat\Omega$, $\tilde{\mathcal{G}} := \mathcal{F}\otimes\hat{\mathcal{F}}$ and $\tilde{Q} := P\times\hat{P}$. Then let $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{P})$ be the completion of $(\tilde\Omega, \tilde{\mathcal{G}}, \tilde{Q})$. Also let $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ be the usual argumentation of $(\tilde{\mathcal{G}}_t)_{t\geq 0}$, where $\tilde{\mathcal{G}}_t := \mathcal{F}_t \otimes \hat{\mathcal{F}}_t$. More specifically, for $t\geq 0$ $\tilde{\mathcal{F}}_t = \bigcap_{s>t}(\tilde{\mathcal{G}}_s \cup \mathcal{N})$, where $\mathcal{N}$ is the collection of null sets of $\tilde{\mathcal{F}}$ under measure $\tilde{P}$. Now the stochastic basis $(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{P})$ satisfies the usual condition. Extend $M.$ and $W.$ to $\tilde\Omega$ via the definition: for each $\tilde\omega := (\omega, \hat\omega) \in \tilde\Omega = \Omega\times\hat\Omega$, $t\in[0,+\infty)$, define $M_t(\omega,\hat\omega) := M_t(\omega)$; $W_t(\omega,\hat\omega) = W_t(\hat\omega)$. Then we have that $(W_t)_{t\geq 0}$ is a Brownian motion on $(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{P})$ independent of $(M_t)_{t\geq 0}$. Define for $t\geq 0$ that $B_t := \int_0^t \mathbb{1}_{\{a(s)>0\}}\frac{1}{\sqrt{a(s)}}\,dM_s + \int_0^t \mathbb{1}_{\{a(s)=0\}}\,dW_s$, which means that $(B_t)_{t\geq 0}$ is a c.l.m. on $(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{P})$. And for each $t\geq 0$ $\langle B\rangle_t = \int_0^t \mathbb{1}_{\{a(s)>0\}}\cdot\frac{1}{a(s)}\,d\langle M\rangle_s$ $\boxed{\text{(Note that } \langle M, W\rangle_t \equiv 0)}$ $+ \int_0^t \mathbb{1}_{\{a(s)=0\}}\,d\langle W\rangle_s = \int_0^t \mathbb{1}_{\{a(s)>0\}}\cdot\frac{a(s)}{a(s)}\,ds + \int_0^t \mathbb{1}_{\{a(s)=0\}}\,ds = \int_0^t 1\,ds = t$.
By Lévy's characterization, $(B_t)_{t\geq 0}$ is also a standard B.M. on $(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{P})$.
Also for $t\geq 0$, $\int_0^t \sqrt{a(s)}\,dB_s = \int_0^t \sqrt{a(s)}\mathbb{1}_{\{a(s)>0\}}\,dB_s + \int_0^t \sqrt{a(s)}\mathbb{1}_{\{a(s)=0\}}\,dB_s = \int_0^t \sqrt{a(s)}\mathbb{1}_{\{a(s)>0\}}\cdot\frac{1}{\sqrt{a(s)}}\,dM_s + \int_0^t \sqrt{a(s)}\mathbb{1}_{\{a(s)>0\}\cap\{a(s)=0\}}\,dW_s = \int_0^t \mathbb{1}_{\{a(s)>0\}}\,dM_s + 0$. To finish the proof, it suffices to show that $\int_0^t \mathbb{1}_{\{a(s)=0\}}\,dM_s = 0$ for $t\geq 0$. Actually since this process is

a c.l.m. and $\left\langle \int_0^\cdot 1_{\{a(s)=0\}} dM_s \right\rangle_t = \int_0^t 1_{\{a(s)=0\}} d\langle M\rangle_s = \int_0^t 1_{\{a(s)=0\}} a(s) ds = 0$ for $t \geq 0$, then the proof is done by Prop 5.7.1) of the NOTE or Prop 4.12 of GTM 274. $\square$

Exercise 2 Suppose that $u(t,x) \in \mathcal{C}([0,T] \times \mathbb{R}) \cap \mathcal{C}^{1,2}((0,T] \times \mathbb{R})$ solves the heat equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\partial_{xx} u(t,x) - k(t,x)u(t,x), & (t,x) \in (0,T] \times \mathbb{R}, \\ u(0,x) = f(x), & x \in \mathbb{R}, \end{cases}$$

where $k(t,x)$ is a bounded continuous function. Suppose that u satisfies the growth condition

$$\sup_{0\le s\le T} |u(s,x)| \le M e^{a|x|^2}$$

for some $0 < a < \frac{1}{2T}$ and $M > 0$. Show that u admits the Feynman–Kac representation

$$u(t,x) = \mathsf{E}^x f(B_t) e^{-\int_0^t k(t-s, B_s)\, ds}.$$

Hint: consider $Y_s = u(t-s, B_s)e^{-\int_0^s k(t-\theta, B_\theta)\, d\theta}$; apply the Optional Sampling Theorem with respect to the stopping times $\tau_n = \inf\{s \ge 0 : |B_s| \ge n\}$ and pass to the limit carefully.

Proof. Define for $t \in [0,T]$ $Z_t := e^{-\int_0^t k(T-\theta, B_\theta) d\theta}$ and $Y_t := u(T-t, B_t) Z_t$. Since the function $(t,x) \mapsto u(T-t,x)$ is $C^{1,2}([0,T), \mathbb{R})$, then by Itô's formula we obtain

$$du(T-t,B_t) = -\partial_t u(T-t,B_t)dt + \partial_x u(T-t,B_t)dB_t + \tfrac{1}{2}\partial_{xx}u(T-t,B_t)dt$$
$$= (-\partial_t u + \tfrac{1}{2}\partial_{xx}u)(T-t,B_t)dt + \partial_x u(T-t,B_t)dB_t.$$

By integration by parts, we have that for $t \in [0,T)$, $Y_t - Y_0$

$$= \int_0^t Z_s \partial_x u(T-s,B_s)dB_s + \int_0^t Z_s(-\partial_t u + \tfrac{1}{2}\partial_{xx}u)(T-s,B_s)ds$$
$$+ \int_0^t u(T-s,B_s) Z_s(-k(T-s,B_s))ds + 0.$$

Since $-\partial_t u + \frac{1}{2}\partial_{xx}u - ku = 0$ in $(0,T] \times \mathbb{R}$, thus we have for $t \in [0,T)$, $Y_t - Y_0 = \int_0^t Z_s \partial_x u(T-s,B_s)dB_s$, which implies that $(Y_t)_{t\in[0,T)}$ is a c.l.m. For $n \in \mathbb{N}^*$ set $\tau_n := \inf\{s \in [0,T), |B_s| \geq n\}$. Then for $t \in [0,T)$ $0 \leq Y_{t\wedge\tau_n} \leq \sup_{s\in[0,T], |x|\leq n} |u(s,x)| \cdot e^{T\cdot\sup|k|} \leq Me^{an^2 + T\sup|k|} < \infty$, which implies $(Y_{t\wedge\tau_n})_{t\in[0,T)}$ is a uniformly integrable martingale. Therefore for $t \in [0,T)$, $x \in \mathbb{R}$,

$$u(T,x) = \mathbb{E}^x[u(T-0,B_0)e^0] = \mathbb{E}^x Y_0 = \mathbb{E}^x Y_{0\wedge\tau_n} = \mathbb{E}^x Y_{t\wedge\tau_n}$$
$$= \mathbb{E}^x[u(T-t\wedge\tau_n, B_{t\wedge\tau_n}) Z_{t\wedge\tau_n}].$$

For each fixed $n \in \mathbb{N}^*$, we know that $Z_{t\wedge\tau_n}$ is bounded in $t \in [0,T)$ since k is bounded. And $|u(T-t\wedge\tau_n, B_{t\wedge\tau_n})| \leq Me^{a|B_{t\wedge\tau_n}|^2} \leq Me^{an^2} < \infty$ for all $t \in [0,T)$. Letting $t \to T$ and by D.C.T. we obtain that for each $n \in \mathbb{N}^*$:

$u(T,x) = \mathbb{E}^x[f(B_{T\wedge\tau_n}) Z_{T\wedge\tau_n}]$ (*) And for each $n \in \mathbb{N}^*$, $|f(B_{T\wedge\tau_n})| \leq M e^{a|B_{T\wedge\tau_n}|^2}$

$\leq M \cdot \sup_{s\in[0,T]} e^{a|B_s|^2}$. Since $(e^{a|B_s|^2})_{s\in[0,T]}$ is an nonnegative submartingale, then by Doob's L^p inequality, $\mathbb{E}\left[\sup_{s\in[0,T]} e^{a|B_s|^2}\right] \leq \frac{e}{e-1} \mathbb{E}\left[e^{a|B_T|^2} \log^+ e^{a|B_T|^2}\right] = \frac{ea}{e-1} \mathbb{E}\left[|B_T|^2 e^{a|B_T|^2}\right]$

$\leq \frac{ea}{e-1} \mathbb{E}\left[|B_T|^4\right]^{\frac{1}{2}} \cdot \mathbb{E}\left[e^{2a|B_T|^2}\right]^{\frac{1}{2}}$. Since $\mathbb{E}|B_T|^4 < \infty$, and

$\mathbb{E}\left[e^{2a|B_T|^2}\right] = \int_{\mathbb{R}} e^{2ax^2} \cdot \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx = \frac{1}{\sqrt{2\pi T}} \int_{\mathbb{R}} e^{-(\frac{1}{2T} - 2a)x^2} dx < \infty$. (since $0 < a < \frac{1}{2T} < \frac{1}{T}$.)

Thus the proof is done by letting $n \to +\infty$ in (*) and D.C.T.