HW11

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Exercise 1 Let $(M_t)_{t\geq 0}$ be a c.l.m. on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ with

$$\langle M\rangle_t = \int_0^t a(s)\, ds$$

for some progressively measurable process $a \geq 0$. Show that there exists an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathsf{P}})$ and a standard Brownian motion B on it such that

$$M_t = \int_0^t \sqrt{a(s)}\, dB_s.$$

Hint: consider

$$B_t := \int_0^t \mathbb{1}_{\{a(s)>0\}} \frac{1}{\sqrt{a(s)}}\, dM_s + \int_0^t \mathbb{1}_{\{a(s)=0\}}\, dW_s$$

where W is a standard Brownian motion independent of everything else.

Exercise 2 Suppose that $u(t,x) \in \mathcal{C}([0,t]\times\mathbb{R}) \cap \mathcal{C}^{1,2}\big((0,t]\times\mathbb{R}\big)$ solves the heat equation

$$\begin{cases} \partial_t u(t,x) = \dfrac{1}{2}\partial_{xx} u(t,x) - k(t,x)u(t,x), & (t,x)\in(0,t]\times\mathbb{R}, \\ u(0,x) = f(x), & x\in\mathbb{R}, \end{cases}$$

where $k(t,x)$ is a bounded continuous function. Suppose that u satisfies the growth condition

$$\sup_{0\leq s\leq t} |u(s,x)| \leq M e^{a|x|^2}$$

for some $0 < a < \dfrac{1}{2t}$ and $M > 0$. Show that u admits the Feynman–Kac representation

$$u(t,x) = \mathsf{E}^x f(B_t) e^{-\int_0^t} k(t-s, B_s)\, ds.$$

Hint: consider $Y_s = u(t-s, B_s)e^{-\int_0^s k(t-\theta, B_\theta)\, d\theta}$; apply the Optional Sampling Theorem with respect to the stopping times $\tau_n = \inf\{s \geq 0 : |B_s| \geq n\}$ and pass to the limit carefully.