HW10

April 23, 2024

Exercise 1 Consider the one-dimensional SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)\, dB_t$$

where b, σ satisfy:

- b is bounded, measurable, and

  $$|b(t,x) - b(t,y)| \le g(|x-y|)$$

  for some continuous, strictly increasing, concave function $g : \mathbb{R}_+ \to \mathbb{R}_+$ with $g(0) = 0$ and $\displaystyle\int_0^1 \frac{du}{g(u)} = \infty$.

- σ is bounded, measurable and

  $$|\sigma(t,x) - \sigma(t,y)| \le h(|x-y|)$$

  for some continuous, strictly increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $h(0) = 0$ and $\displaystyle\int_0^1 \frac{du}{h^2(u)} = \infty$.

Show that pathwise uniqueness holds for this SDE.

Hint: You may use the following result: for g given above, if f is a non-negative continuous function, then

$$f(t) \le \int_0^t g\big(f(s)\big)\, ds, \quad t \ge 0 \qquad \Rightarrow \qquad f(t) \equiv 0, \quad t \ge 0.$$

If $X^{(j)}$, $j = 1, 2$ are two weak solutions, you are aiming at $f(t) = \mathsf{E}|X_t^{(1)} - X_t^{(2)}|$ satisfying the above integral inequality.

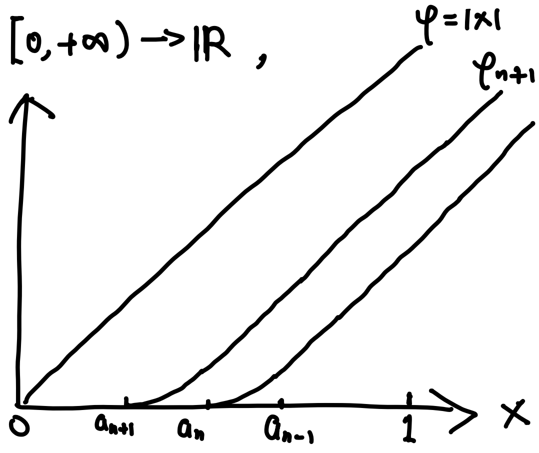
proof. Suppose $(X^{(1)}, B)$ and $(X^{(2)}, B)$ are two weak solutions on a probability space with a same initial condition. Then we have for $t \geq 0$

$$X_t^{(1)} - X_t^{(2)} = \int_0^t [b(s, X_s^{(1)}) - b(s, X_s^{(2)})]\, ds + \int_0^t [\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})]\, dB_s \quad (*)$$

By condition $\int_0^1 \frac{du}{h^2(u)} = +\infty$, there exists a decreasing sequence $1 = a_0 > a_1 > \cdots > a_n > \cdots > 0$ such that $a_n \searrow 0$ and for $n \in \mathbb{N}^*$ $\int_{a_n}^{a_{n-1}} h^{-2}(u)\,du = n$, which means $\int_{a_n}^{a_{n-1}} \frac{h^{-2}(u)}{n}\,du = 1$.

Let us consider a sequence of functions, $\varphi_n : [0, +\infty) \to \mathbb{R}$,

$$x \mapsto \begin{cases} 0, & 0 \leq x \leq a_n, \\ \int_0^x \int_0^t \frac{h^{-2}(u)}{n}\, du\, dt, & a_n < x \leq a_{n-1} \\ x - a_{n-1} + \varphi(a_{n-1}), & x > a_{n-1}. \end{cases}$$



And extend φ_n to $(-\infty, 0)$ by setting $\varphi(x) = \varphi(-x)$.

It could be observed from the construction of φ_n that $\varphi_n(x) \nearrow |x|$ as $n \to +\infty$ pointwise on $\mathbb{R}$. Also we know that $\varphi_n \in C^2(\mathbb{R})$ for $n \in \mathbb{N}^*$. By applying φ_n and Itô's formula to equation $(*)$, we have that for each $n \in \mathbb{N}^*$, $t \geq 0$:

$$\varphi_n(X_t^{(1)} - X_t^{(2)}) = \int_0^t \varphi_n'(X_s^{(1)} - X_s^{(2)})[b(s, X_s^{(1)}) - b(s, X_s^{(2)})]ds$$

$$+ \int_0^t \varphi_n'(X_s^{(1)} - X_s^{(2)})[\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})]dB_s$$

$$+ \int_0^t \frac{1}{2}\varphi_n''(X_s^{(1)} - X_s^{(2)})[\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})]^2 ds =: I_1^{(n)}(t) + I_2^{(n)}(t) + I_3^{(n)}(t),$$

where $I_1^{(n)}(t) := \int_0^t \varphi_n'(X_s^{(1)} - X_s^{(2)})[b(s, X_s^{(1)}) - b(s, X_s^{(2)})]ds$,

$I_2^{(n)}(t) := \int_0^t \varphi_n'(X_s^{(1)} - X_s^{(2)})[\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})]dB_s$ and

$I_3^{(n)}(t) := \int_0^t \frac{1}{2}\varphi_n''(X_s^{(1)} - X_s^{(2)})[\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})]^2 ds$. Note that $\mathbb{E} I_2(t) = 0$, and for $n \in \mathbb{N}^*$:

$$|I_1^{(n)}(t)| \leq \int_0^t \varphi_n'(X_s^{(1)} - X_s^{(2)})|b(s, X_s^{(1)}) - b(s, X_s^{(2)})| ds \leq \int_0^t \max_{s \geq 0} \varphi_n'(s) \cdot g(|X_s^{(1)} - X_s^{(2)}|) ds$$

$$= \int_0^t g(|X_s^{(1)} - X_s^{(2)}|) ds.$$

And $|I_3^{(n)}(t)| = \int_0^t \frac{1}{2} \cdot \frac{1}{n} h^{-2}(X_s^{(1)} - X_s^{(2)})[\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})]^2 ds$

$$\leq \int_0^t \frac{1}{2} \cdot \frac{1}{n} h^{-2}(X_s^{(1)} - X_s^{(2)}) h^2(|X_s^{(1)} - X_s^{(2)}|) ds$$

$$\leq \frac{1}{2} t \cdot \max_{u \in [0,t]} \frac{h^{-2}(u)}{n} h^2(|u|) ds = \frac{1}{2} t \cdot \max_{u \in [0,t]} \frac{h^{-2}(|u|)}{n} h^2(u) ds = \frac{1}{2} \cdot \frac{t}{n}.$$

Thus we have that $\mathbb{E}\varphi_n(X_t^{(1)} - X_t^{(2)}) = \mathbb{E} I_1^{(n)}(t) + \mathbb{E} I_3^{(n)}(t) \leq \mathbb{E}\int_0^t g(|X_s^{(1)} - X_s^{(2)}|) ds + \frac{1}{2} \cdot \frac{t}{n} \underset{\text{(Fubini)}}{=} \int_0^t \mathbb{E} g(|X_s^{(1)} - X_s^{(2)}|) ds + \frac{1}{2} \frac{t}{n} \underset{\text{(Jensen)}}{\leq} \int_0^t g(\mathbb{E}|X_s^{(1)} - X_s^{(2)}|) ds + \frac{1}{2} \cdot \frac{t}{n}$.

Letting $n \to +\infty$ and by M.C.T. we obtain that for $t \geq 0$:

$\mathbb{E}|X_t^{(1)} - X_t^{(2)}| \leq \int_0^t g(\mathbb{E}|X_s^{(1)} - X_s^{(2)}|) ds$, then the proof is done by continuity of $X^{(1)}, X^{(2)}$ and the result provided by the hint above.

Exercise 2 Let $d \geq 2$ and let $W = (W_t^{(1)}, \cdots, W_t^{(d)})$ be a d-dimensional Brownian motion starting from $x \neq 0$. Let $\tau = \inf\{t : |W_t| = 0\}$. Recall that $\mathsf{P}^x(\tau = \infty) = 1$.

1. Show that $(|W_t|, \beta_t)$ where $\beta_t = \sum_{i=1}^{d} \int_0^t \frac{W_s^{(i)}}{|W_s|} \, dW_s^{(i)} \cancel{ds}$ is a weak solution to

$$dX_t = \frac{d-1}{2} \frac{dt}{X_t} + dB_t, \quad X_0 = |x| > 0. \tag{1}$$

proof. By Ex 5.33. 2) we know that $(\beta_t)_{t\geq 0}$ is a Brownian motion. And by Ex 5.33. 6) we know that $d|W_t| = \frac{d-1}{2} \frac{dt}{|W_t|} + d\beta_t$, $|W_0| = |x|$. Thus $(|W|, \beta)$ is a weak solution to (1).

2. Show that if (X_t, B_t) is a weak solution to Eq. (1), then $(|X_t|^2, B_t)$ is a weak solution to

$$dY_t = 2\sqrt{|Y_t|}\, dB_t + d \cdot dt \tag{2}$$

proof. By Itô's formula, $d|X_t|^2 = 2X_t\, dX_t + d\langle X \rangle_t$ $= 2X_t \cdot \frac{d-1}{2} \cdot \frac{dt}{X_t} + 2X_t\, dB_t + dt = 2X_t\, dB_t + d \cdot dt$. And we know that $X_t > 0$ for all $t \geq 0$ since $X_0 = |x| > 0$ and if there exists $t_0 > 0$ s.t. $X_{t_0} < 0$ then there must exists $t_1 \in [0, t_0)$ s.t. $X_{t_1} = 0$ which is impossible. Thus we have $\sqrt{|X_t|^2} = |X_t| = X_t$ for $t \geq 0$. Then the proof is done.

3. Explain why pathwise uniqueness holds for Eq. (1) and Eq. (2).

proof. Apply Proposition 9.8 to $b(t,x) = \frac{d-1}{2} \cdot \frac{1}{x}$, we obtain the pathwise uniqueness of Eq.(1). For Eq.(2), note that $b(t,x) = d$ satisfies the global Lipschitz condition. And set $\sigma(t,x) = 2\sqrt{|x|}$ then $2\left|\sqrt{|x|} - \sqrt{|y|}\right| \leq 2\sqrt{|x-y|}$ for $x, y \in \mathbb{R}$. And let $h(x) := 2\sqrt{x}$ then $\int_0^1 h^{-2}(u)\,du = \int_0^1 \frac{1}{4u}\,du = +\infty$. Thus the pathwise uniqueness of Eq (2) is obtained by Corollary 9.10 or Ex 1 above.

Exercise 3 Let $b_{1,2}(t,x), a(t,x) : \mathbb{R}_+ \times \mathbb{R}^2$ be locally bounded measurable functions. Let $X_t = (X_t^{(1)}, X_t^{(2)})$ be a progressively measurable process in $\mathbb{R}^2$, such that

$$M_t^{(j)} = X_t^{(j)} - X_0^{(j)} - \int_0^t b_j(s, X_s)\, ds, \quad j = 1, 2,$$

and

$$M_t^{(12)} = X_t^{(1)} X_t^{(2)} - X_0^{(1)} X_0^{(2)} - \int_0^t \left[X_s^{(1)} b_2(s, X_s) + X_s^{(2)} b_1(s, X_s) + a(s, X_s) \right] ds$$

are all c.l.m.'s.

Show that

$$\langle M^{(1)}, M^{(2)} \rangle_t = \int_0^t a(s, X_s)\, ds.$$

proof. For $t \geq 0$, $M_t^{(1)} M_t^{(2)} - \int_0^t a(s,X_s)\,ds = X_t^{(1)} X_t^{(2)} - X_t^{(1)} X_0^{(2)} - X_t^{(1)} \int_0^t b_2(s,X_s)\,ds$
$- X_0^{(1)} X_t^{(2)} + X_0^{(1)} X_0^{(2)} + X_0^{(1)} \int_0^t b_2(s,X_s)\,ds - X_t^{(2)} \int_0^t b_1(s,X_s)\,ds + X_0^{(2)} \int_0^t b_1(s,X_s)\,ds$
$+ \int_0^t b_1(s,X_s)\,ds \int_0^t b_2(s,X_s)\,ds - \int_0^t a(s,X_s)\,ds$
$= X_t^{(1)} X_t^{(2)} - X_0^{(1)} X_0^{(2)} - \int_0^t [X_s^{(1)} b_2(s,X_s) + X_s^{(2)} b_1(s,X_s) + a(s,X_s)]\,ds + 2 X_0^{(1)} X_0^{(2)}$
$+ \int_0^t [(X_s^{(1)} - X_t^{(1)}) b_2(s,X_s) + (X_s^{(2)} - X_t^{(2)}) b_1(s,X_s)]\,ds + \int_0^t b_1(s,X_s)\,ds \cdot$
$\int_0^t b_2(s,X_s)\,ds - X_0^{(2)} \cdot \left[X_t^{(1)} - \int_0^t b_1(s,X_s)\,ds\right] - X_0^{(1)} \left[X_t^{(2)} - \int_0^t b_2(s,X_s)\,ds\right]$
$= M_t^{(12)} - X_0^{(2)} M_t^{(1)} - X_0^{(1)} M_t^{(2)} + \tilde{M}_t$, where

$$\tilde{M}_t := \int_0^t (X_s^{(1)} - X_t^{(1)}) b_2(s,X_s)\,ds + \int_0^t (X_s^{(2)} - X_t^{(2)}) b_1(s,X_s)\,ds$$
$$+ \int_0^t b_1(s,X_s)\,ds \cdot \int_0^t b_2(s,X_s)\,ds.$$

By condition, it suffices to show that $(\tilde{M}_t)_{t \geq 0}$ is a c.l.m.

But we have that $\tilde{M}_t = \int_0^t (M_s^{(1)} - M_t^{(1)} - \int_s^t b_1(\tau, X_\tau)\,d\tau) b_2(s,X_s)\,ds$
$+ \int_0^t (M_s^{(2)} - M_t^{(2)} - \int_s^t b_2(\tau,X_\tau)\,d\tau) b_1(s,X_s)\,ds = \int_0^t (M_s^{(1)} - M_t^{(1)}) b_2(s,X_s)\,ds$
$+ \int_0^t (M_s^{(2)} - M_t^{(2)}) b_1(s,X_s)\,ds - \int_0^t \int_s^t b_1(\tau,X_\tau)\,d\tau\, b_2(s,X_s)\,ds$
$- \int_0^t \int_s^t b_2(\tau,X_\tau)\,d\tau\, b_1(s,X_s)\,ds + \int_0^t b_1(s,X_s)\,ds \int_0^t b_2(s,X_s)\,ds.$

Note that $\int_0^t \int_s^t b_1(\tau,X_\tau)\,d\tau\, b_2(s,X_s)\,ds = \int_\cdot^t b_1(\tau,X_\tau)\,d\tau \cdot \int_0^\cdot b_2(s,X_s)\,ds \Big|_0^t$ $-$

$\int_0^t -b_1(s,X_s)\cdot\int_0^s b_2(\tau,X_\tau)d\tau\, ds = 0 + \int_0^t b_1(s,X_s)\int_0^s b_2(\tau,X_\tau)d\tau\, ds$. Therefore we have that $\int_0^t\int_s^t b_1(\tau,X_\tau)d\tau\, b_2(s,X_s)ds + \int_0^t\int_s^t b_2(\tau,X_\tau)d\tau\, b_1(s,X_s)ds$
$= \int_0^t b_1(s,X_s)\int_0^s b_2(\tau,X_\tau)d\tau\, ds + \int_0^t b_1(s,X_s)\int_s^t b_2(\tau,X_\tau)d\tau\, ds$
$= \int_0^t b_1(s,X_s)ds\cdot\int_0^t b_2(s,X_s)ds$. Therefore we obtain that
$\tilde{M}_t = \int_0^t (M_s^{(1)} - M_t^{(1)})b_2(s,X_s)ds + \int_0^t (M_s^{(2)} - M_t^{(2)})b_1(s,X_s)ds$. By Itô's formula, we have that for $t\geq 0$, $M_t^{(1)}\int_0^t b_2(s,X_s)ds = \int_0^t\int_0^s b_2(\tau,X_\tau)d\tau\, dM_s^{(1)}$
$+ \int_0^t M_s^{(1)} b_2(s,X_s)ds$. Similarly, $M_t^{(2)}\int_0^t b_1(s,X_s)ds = \int_0^t\int_0^s b_1(\tau,X_\tau)d\tau\, dM_s^{(2)}$
$+ \int_0^t M_s^{(2)} b_1(s,X_s)ds$. Therefore we obtain that
$\tilde{M}_t = -\int_0^t\int_0^s b_2(\tau,X_\tau)d\tau\, dM_s^{(1)} - \int_0^t\int_0^s b_1(\tau,X_\tau)d\tau\, dM_s^{(2)}$, which is obviously a continuous local martingale. Thus $\left(M^{(1)}M_t^{(2)} - \int_0^t a(s,X_s)ds\right)_{t\geq 0}$ is a c.l.m., which implies the result desired.