HW10

April 23, 2024

Exercise 1 Consider the one-dimensional SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)\, dB_t$$

where b, σ satisfy:

- b is bounded, measurable, and

$$|b(t, x) - b(t, y)| \leq g(|x - y|)$$

for some continuous, strictly increasing, concave function $g : \mathbb{R}_+ \to \mathbb{R}_+$ with $g(0) = 0$ and $\displaystyle\int_0^1 \frac{du}{g(u)} = \infty$.

- σ is bounded, measurable and

$$|\sigma(t, x) - \sigma(t, y)| \leq h(|x - y|)$$

for some continuous, strictly increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $h(0) = 0$ and $\displaystyle\int_0^1 \frac{du}{h^2(u)} = \infty$.

Show that pathwise uniqueness holds for this SDE.

Hint: You may use the following result: for g given above, if f is a non-negative continuous function, then

$$f(t) \leq \int_0^t g\big(f(s)\big)\, ds, \quad t \geq 0 \qquad \Rightarrow \qquad f(t) \equiv 0, \quad t \geq 0.$$

If $X^{(j)}$, $j = 1, 2$ are two weak solutions, you are aiming at $f(t) = \mathsf{E}|X_t^{(1)} - X_t^{(2)}|$ satisfying the above integral inequality.

Exercise 2 Let $d \geq 2$ and let $W = (W_t^{(1)}, \cdots, W_t^{(d)})$ be a d-dimensional Brownian motion starting from $x \neq 0$. Let $\tau = \inf\{t : |W_t| = 0\}$. Recall that $\mathsf{P}^x(\tau = \infty) = 1$.

1. Show that $(|W_t|, \beta_t)$ where $\beta_t = \displaystyle\sum_{i=1}^d \int_0^t \frac{W_s^{(i)}}{|W_s|}\, ds$ is a weak solution to

$$dX_t = \frac{d-1}{2} \frac{dt}{X_t} + dB_t, \quad X_0 = |x| > 0. \tag{1}$$

2. Show that if (X_t, B_t) is a weak solution to Eq. (1), then $(|X_t|^2, B_t)$ is a weak solution to

$$dY_t = 2\sqrt{|Y_t|}\, dB_t + d \cdot dt \tag{2}$$

3. Explain why pathwise uniqueness holds for Eq. (1) and Eq. (2).

Exercise 3 Let $b_{1,2}(t,x), a(t,x) : \mathbb{R}_+ \times \mathbb{R}^2$ be locally bounded measurable functions. Let $X_t = (X_t^{(1)}, X_t^{(2)})$ be a progressively measurable process in $\mathbb{R}^2$, such that

$$M_t^{(j)} = X_t^{(j)} - X_0^{(j)} - \int_0^t b_j(s, X_s)\, ds, \quad j = 1, 2,$$

and

$$M_t^{(12)} = X_t^{(1)} X_t^{(2)} - X_0^{(1)} X_0^{(2)} - \int_0^t \Big[X_s^{(1)} b_2(s, X_s) + X_s^{(2)} b_1(s, X_s) + a(s, X_s) \Big]\, ds$$

are all c.l.m.'s.

Show that

$$\langle M^{(1)}, M^{(2)} \rangle_t = \int_0^t a(s, X_s)\, ds.$$