HW1

February 19, 2024

Exercise 1 (Transformation of BM)

1. Prove the equivalency of the following two conditions: for $0 = t_0 \le t_1 < \cdots < t_m$,

$$\begin{aligned} &\mathcal{L}(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}) = \mathcal{N}(0, \mathrm{diag}\{t_{i+1} - t_i\}_{0 \le i \le m-1}) \\ \Leftrightarrow \quad &(B_{t_1}, B_{t_2}, \ldots, B_{t_m}) \text{ is a centered Gaussian vector with covariance } \mathsf{E} B_{t_i} B_{t_j} = t_i \wedge t_j. \end{aligned} \tag{1}$$

2. Suppose that $(B_t)_{t\ge0}$ has f.d.d. Eq. (1). Show that all the following processes have the same f.d.d. Eq. (1).
   a) $(-B_t)_{t\ge0}$.
   b) $(B_t^{\lambda})_{t\ge0} := (\frac{1}{\lambda} B_{\lambda^2 t})_{t\ge0}$. (Fix $\lambda > 0$.)
   c) $(B_t^{(s)})_{t\ge0} := (B_{t+s} - B_s)_{t\ge0}$. (Fix $s > 0$.)
   d) $(tB_{1/t})_{t\ge0}$ (with the convention $0 \cdot B_{1/0} = 0$).

Hint: You can find some basic properties of Gaussian vectors in Section 2.1. This exercise is basically about covariance computation.

Exercise 2 Let $(X_n)_{n\ge1}$ and X_∞ be r.v.'s on $(\Omega, \mathcal{F}, \mathsf{P})$. Show that

$$\{\omega : \lim_{n\to\infty} X_n(\omega) = X_\infty(\omega)\} = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega : |X_n(\omega) - X_\infty(\omega)| < \frac{1}{m}\}$$

Conclude that the left hand side belongs to $\mathcal{F}$.

Exercise 3 Let $X = (X_t)_{t\ge0}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathsf{P})$ such that $t \mapsto X_t(\omega)$ is continuous for almost every $\omega \in \Omega$. Let τ be a continuous r.v. on $(\Omega, \mathcal{F}, \mathsf{P})$ and $Y = (Y_t)_{t\ge0}$ be defined as

$$Y_t(\omega) = \begin{cases} X_t(\omega), & t \ne \tau(\omega), \\ X_t(\omega) + 1, & t = \tau(\omega). \end{cases}$$

Show that Y is a stochastic process which is a modification of X, but $t \mapsto Y_t(\omega)$ is NOT continuous for almost every $\omega \in \Omega$.

Exercise 4 Use the Abel transformation (summation by parts)

$$\sum_{k=1}^{n} u_k(v_{k+1} - v_k) = u_{n+1}v_{n+1} - u_1 v_1 - \sum_{k=1}^{n} v_{k+1}(u_{k+1} - u_k)$$

to show that integration by parts holds for Riemann–Stieltjes integrals for functions f and g of bounded variation.