HW5

Total Points: 100

1. [10 points] Let $\Omega = \{(x_1, x_2) : x_1, x_2 > 0\}$ be the first quadrant. Use reflection symmetry to construct a Green's function in Ω, i.e., for each $y \in \Omega$, solve

$$\begin{cases} -\Delta G(x) = \delta(x-y), & x \in \Omega, \\ G(x) = 0, & x \in \partial\Omega. \end{cases}$$

[Total for Question 1: 10 points]

2. [10 points] Let Ω be a bounded domain with $\mathcal{C}^1$-boundary. Show that if there exists a solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$ to the Neumann problem

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = g, & x \in \partial\Omega, \end{cases}$$

where f, g are bounded, then

$$\int_\Omega f \, dx = -\int_{\partial\Omega} g \, dS.$$

Hint: use integration by parts on $\int_\Omega (\Delta u) v \, dx$ *with* $v \equiv 1$.

[Total for Question 2: 10 points]

3. [20 points] For $\Omega = B_r(0) \subset \mathbb{R}^2$, let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases}$$

where f, g are bounded and continuous. Show that u satisfies

$$u(0) = \frac{1}{2\pi r} \int_{\partial\Omega} g(x) \, dS(x) + \frac{1}{2\pi} \int_\Omega (\ln r - \ln|x|) f(x) \, dx.$$

Hint: consider

$$\varphi(t) = \frac{1}{2\pi t} \int_{\partial B_t(0)} u(x) \, dS(x) + \frac{1}{2\pi} \int_{B_t(0)} (\ln t - \ln|x|) f(x) \, dx,$$

and show that $\varphi'(t) = 0$, $\lim_{t \to 0+} \varphi(t) = u(0)$.

[Total for Question 3: 20 points]

4. Let $v \in \mathcal{C}^2(\Omega)$. We say that v is subharmonic in Ω if $-\Delta v \leq 0$ in Ω.

   (a) [10 points] Show that if v is subharmonic, then for any $\overline{B_r(x)} \subset \Omega$,

   $$v(x) \leq \fint_{B_r(x)} v(y)\, dy.$$

   Hint: let $\varphi(r) = \fint_{B_r(x)} v(y)\, dy$ *and consider* $\varphi'(r)$.

   (b) [5 points] Let Ω be a bounded domain. Show that if $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and v is subharmonic, then

   $$\max_{\bar{\Omega}} v(x) = \max_{\partial\Omega} v(x).$$

   (c) [5 points] Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth convex function. Show that if u is harmonic, then $v = \phi(u)$ is subharmonic.

[Total for Question 4: 20 points]

5. [20 points] Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $u(x) \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u = 1, & \Omega, \\ u = 0, & \partial\Omega. \end{cases}$$

Show that for any $x_0 \in \Omega$,

$$\frac{1}{2d} \min_{x \in \partial\Omega} |x - x_0|^2 \leq u(x_0) \leq \frac{1}{2d} \max_{x \in \partial\Omega} |x - x_0|^2.$$

Hint: apply weak maximum principle to $v(x) = u(x) + \frac{1}{2d}|x - x_0|^2$.

[Total for Question 5: 20 points]

6. [10 points] Let u be harmonic in $\mathbb{R}^d$, $d \geq 2$. Suppose there exists a constant $C > 0$ such that

$$|u(x)| \leq C(1 + |x|), \quad x \in \mathbb{R}^d.$$

Show that $\partial_{ij} u \equiv 0$ and hence $u(x) = a \cdot x + b$ for some $a \in \mathbb{R}^d$, $b \in \mathbb{R}$.

[Total for Question 6: 10 points]

7. [10 points] Let Ω be a bounded domain with $\mathcal{C}^1$-boundary. Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u + c(x)u = f(x), & \Omega, \\ \dfrac{\partial u}{\partial n} + \alpha(x)u = 0, & \partial\Omega, \end{cases}$$

where $\alpha(x) \geq 0$ and $c(x) \geq c_0 > 0$. Assume that $f \in L^2(\Omega)$.

Show that there exists a constant $M = M(c_0)$,

$$\int_\Omega |\nabla u(x)|^2\, dx + \frac{c_0}{2} \int_\Omega |u(x)|^2\, dx + \int_{\partial\Omega} \alpha(x) u^2(x)\, dS(x) \leq M \int_\Omega |f(x)|^2\, dx.$$

[Total for Question 7: 10 points]