

HW4

Total Points: 100

1. Consider the following parabolic operator:

$$\mathcal{L}_t h := \sum_{i,j=1}^d a_{i,j}(t, x) h_{x_i x_j} + \sum_{i=1}^d b_i(t, x) h_{x_i} - c(t, x) h,$$

where $A = (a_{ij})$ is symmetric positive semidefinite for every (t, x) . Let Ω be a bounded domain.

- (a) [10 points] Let $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$. Assume that $(\partial_t - \mathcal{L}_t)u \leq 0$ in Ω_T . Assume that $c \geq 0$. Show that

$$\max_{\overline{\Omega_T}} u \leq \max(0, \max_{\partial_p \Omega_T} u).$$

- (b) [5 points] Assume that $c(t, x) \geq -K$ where $K > 0$. Show that

$$\max_{\overline{\Omega_t}} u \leq e^{Kt} \max(0, \max_{\partial_p \Omega_t} u).$$

Hint: consider $v(t, x) = e^{-Kt} u(t, x)$.

- (c) [5 points] Assume that $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$ is the classical solution to

$$\begin{cases} u_t = \mathcal{L}_t u + f(t, x), & (t, x) \in \Omega_T, \\ u(t, x) = 0, & t > 0, x \in \partial\Omega, \\ u(0, x) = \phi(x), & x \in \Omega. \end{cases}$$

Show that

$$\sup_{x \in \Omega} |u(t, x)| \leq e^{Kt} \|\phi\|_{L^\infty} + \frac{e^{Kt} - 1}{K} \|f\|_{L^\infty}.$$

Hint: apply maximum principle to $w(t) \pm u(t, x)$ where $w(t)$ is the RHS.

[Total for Question 1: 20 points]

2. Let

$$\mathcal{L}u = \Delta u + \sum_{i=1}^d b_i(x) u_{x_i},$$

where b_i are bounded and continuous. Let $u \in \mathcal{C}^{1,2}((0, \infty) \times \mathbb{R}^d) \cap \mathcal{C}([0, \infty) \times \mathbb{R}^d)$. Assume that $u_t - \mathcal{L}u \leq 0$ for $t > 0$, $u(0, x) \leq 0$, and

$$|u(t, x)| \leq K(1 + |x|^p), \quad 0 \leq p < 2.$$

- (a) [5 points] Show that there exists $C > 0$ such that

$$(\partial_t - \mathcal{L})(e^{Ct}(1 + |x|^2)) \geq 0.$$

- (b) [5 points] Show that $u \leq 0$.

Hint: apply maximum principle on $u_\varepsilon(t, x) = u(t, x) - \varepsilon e^{Ct}(1 + |x|^2)$ on a sufficiently large ball B_R .

[Total for Question 2: 10 points]

3. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a global Lipschitz function with constant L , that is $|F(x) - F(y)| \leq L|x - y|$. Let u, v be classical solutions to $z_t - \Delta z = F(z)$ on a bounded domain Ω with zero Dirichlet boundary condition.

- (a) [5 points] Let $w = u - v$. Show that

$$w_t - \Delta w = c(t, x)w,$$

for some function $|c(t, x)| \leq L$.

- (b) [5 points] Show that

$$\sup_{x \in \Omega} |u(t, x) - v(t, x)| \leq e^{Lt} \sup_{x \in \Omega} |u(0, x) - v(0, x)|.$$

- (c) [5 points] Prove the comparison principle:

$$u(0, x) \leq v(0, x) \implies u(t, x) \leq v(t, x).$$

[Total for Question 3: 15 points]

4. [10 points] Let Ω be a bounded domain and let $u, v \in \mathcal{C}^{1,2}((0, \infty) \times \Omega) \cap \mathcal{C}([0, \infty) \times \overline{\Omega})$ solve

$$u_t + H(\nabla u) = \Delta u, \quad v_t + |\nabla v|^2 = \Delta v,$$

in Ω_T with zero Dirichlet boundary condition, where $H : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies $H(p) \geq |p|^2$.

Assume that $u(0, x) \leq v(0, x)$. Show that

$$u(t, x) \leq v(t, x).$$

Hint: for $T > 0$, consider $(t^, x^*) = \operatorname{argmax}_{\overline{\Omega_T}} (u - v - \varepsilon t)$ and show that $(t^*, x^*) \notin \Omega_T$.*

[Total for Question 4: 10 points]

5. Let $\Omega = (0, \ell)$.

- (a) [10 points] Let $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$ satisfy

$$\begin{cases} u_t - u_{xx} \geq 0, & (t, x) \in \Omega_T, \\ u|_{t=0} \geq 0, & x \in \Omega, \\ u(t, 0) \geq 0, & t > 0, \\ u_x(t, \ell) \geq 0, & t > 0. \end{cases}$$

Show that $u \geq 0$ on $\overline{\Omega_T}$.

(b) [5 points] Let $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$ satisfy

$$\begin{cases} u_t - u_{xx} = f, & (t, x) \in \Omega_T, \\ u|_{t=0} = \varphi, & x \in \Omega, \\ u(t, 0) = 0, & t > 0, \\ u_x(t, \ell) = g(t), & t > 0, \end{cases}$$

where f, φ, g are bounded, continuous functions in their domains. Show that

$$\max_{\overline{\Omega_T}} |u| \leq C(T+1)(F+G+\Phi)$$

for some constant C depending only on ℓ , where $F = \sup|f|$, $G = \sup|g|$ and $\Phi = \sup|\varphi|$.

Hint: consider $v(t, x) = u(t, x) + \varepsilon(1 + t + x)$ and its point of minimum.

[Total for Question 5: 15 points]

6. [10 points] Let $\Omega = (0, \ell)$. Suppose that $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$ solves

$$\begin{cases} u_t - u_{xx} = f(t, x), & (t, x) \in \Omega_T, \\ u(0, x) = 0, & x \in [0, \ell], \\ -u_x + \alpha u = 0, & t > 0, x = 0, \\ u_x + \beta u = 0, & t > 0, x = \ell, \end{cases}$$

where $\alpha, \beta \geq 0$ are constants. Show that

$$\sup_{0 \leq t \leq T} \int_0^\ell u^2(t, x) dx + \int_0^T \int_0^\ell u_x^2(t, x) dx dt \leq C \int_0^T \int_0^\ell f^2(t, x) dx dt,$$

for some constant C depending only on T .

[Total for Question 6: 10 points]

7. [10 points] Let $\Omega = (0, \ell)$ and $b, c \in \mathcal{C}(\overline{\Omega_T})$. Suppose that $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$ solves

$$\begin{cases} u_t - u_{xx} + b(t, x)u_x + c(t, x)u = 0, & (t, x) \in \Omega_T, \\ u(0, x) = \varphi(x), & x \in [0, \ell], \\ u(t, 0) = u(t, \ell) = 0, & t \in [0, T]. \end{cases}$$

Show that

$$\sup_{0 \leq t \leq T} \int_0^\ell u^2(t, x) dx + \int_0^T \int_0^\ell u_x^2(t, x) dx dt \leq C \int_0^\ell \varphi^2(x) dx,$$

for some constant C depending only on T, β and γ , where

$$\beta = \sup_{\overline{\Omega_T}} |b(t, x)|, \quad \gamma = \sup_{\overline{\Omega_T}} |c(t, x)|.$$

[Total for Question 7: 10 points]

8. Let $f \in \mathcal{C}^2(\mathbb{R}^3)$ be supported on $B_1(0)$.

(a) [5 points] Use the fundamental solution in $\mathbb{R}^3$ to find a solution to the equation

$$\begin{cases} -\Delta u(x) = f(x), & x \in \mathbb{R}^3, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases}$$

(You can leave the answer as an integral.)

(b) [5 points] Show that $u(x) = \frac{c}{|x|} + O(|x|^{-2})$ for $|x|$ large, and determine the constant c .

Remark: the physical interpretation is that at large distance, charges supported in a ball look like a point charge.

[Total for Question 8: 10 points]