HW1

Total Points: 100

1. [10 points] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Show that

$$\int_{\partial\Omega} x_i n_j \, dS = \delta_{ij} |\Omega|, \quad i, j = 1, \ldots, n.$$

2. [10 points] Use the method of characteristics to solve the PDE:

$$u_t + 2u_x = 4t, \qquad t > 0, \ x \in \mathbb{R},$$

with initial condition

$$u(0, x) = \sin x.$$

3. [10 points] Use the method of characteristics to solve the PDE:

$$u_t + xu_x = -u, \quad t > 0, \ x \in \mathbb{R},$$

with initial condition

$$u(0, x) = x^2.$$

4. Consider the first-order PDE

$$yu_x + xu_y = u, \quad x, y \in \mathbb{R},$$

with data prescribed on the curve $\Gamma = \{(1, y) : y > 0\}$:

$$u(1, y) = y, \quad y > 0.$$

(a) [10 points] Solve the characteristic system

$$\frac{dx}{ds} = y, \quad \frac{dy}{ds} = x, \quad \big(x(0), y(0)\big) = (1, \eta).$$

(b) [10 points] Determine the region reached by the characteristics in the (x, y)-plane, and solve the equation in this region.
Hint: $x^2 - y^2$ is invariant along the characteristics.

[Total for Question 4: 20 points]

5. [20 points] Consider the inviscid Burgers equation

$$u_t + u \cdot u_x = 0, \quad t > 0,\ x \in \mathbb{R},$$

With initial condition

$$u(0, x) = 1 - x.$$

Find the largest time t_c such that there is no intersection of characteristics on $[0, t_c)$, and find the classical solution for $t \in (0, t_c)$.

6. Consider the damped inviscid Burgers equation

$$u_t + uu_x + u = 0, \quad t > 0,\ x \in \mathbb{R}.$$

with initial condition

$$u(0, x) = 2 \sin x.$$

(a) [10 points] Find the characteristic from every point $(0, \xi)$, $\xi \in \mathbb{R}$.

(b) [10 points] At any $t > 0$, the map $\xi \mapsto x$ is implicit. From the implicit relation, compute

$$\frac{\partial x}{\partial \xi}$$

and determine the first time t_c where characteristics intersect. Find the values of ξ at which the first characteristic crossing occurs.

Hint: for non-intersection you need $\frac{\partial x}{\partial \xi} > 0$.

(c) [10 points] Explain what happens to u_x as $t \to t_c-$, even though u itself remains bounded.

[Total for Question 6: 30 points]