HW9

November 27, 2025

Exercise 1 Prove the following form of Gronwall's inequality.

Let $\eta(t)$ be non-negative and absolutely continuous on $[0,T]$. Assume that the differential inequality

$$\eta'(t) \le \phi(t)\eta(t) + \psi(t), \quad \text{a.e. } t \in [0,T],$$

holds, where ϕ, ψ are non-negative and integrable on $[0,T]$. Show that

$$\eta(t) \le e^{\int_0^t \phi(s)\,ds}\big[\eta(0) + \int_0^t \psi(s)\,ds\big], \quad \forall t \in [0,T].$$

Note: η being absolutely continuous means that $\eta'(t)$ exists for a.e. $t \in [0,T]$ and

$$\eta(t) - \eta(0) = \int_0^t \eta(s)\,ds, \quad \forall t \in [0,T].$$

If you are not comfortable with this part of real analysis, you can do this exercise assuming $\eta \in \mathcal{C}^1$ and ignoring all the "a.e." above.

Exercise 2 Let Ω be a bounded domain and $u \in \mathcal{C}^{1,2}(\overline{\Omega_T})$ solve the equation

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + \displaystyle\sum_{i=1}^d b^i(t,x)\partial_{x_i} u(t,x) = f(t,x), & (t,x) \in \Omega_T, \\ u(t,x) = 0, & x \in \partial\Omega,\ t \ge 0, \\ u(0,x) = g(x), & x \in \bar{\Omega}. \end{cases}$$

Let $b^i(t,x), f \in \mathcal{C}(\overline{\Omega_T})$ and $g \in \mathcal{C}(\bar{\Omega})$.

1. For $u, v \in \mathcal{C}_0^2(\Omega)$, let

   $$B[u,v;t] := \int_\Omega \nabla u \cdot \nabla v + \sum_{i=1}^d b^i \partial_{x_i} u \cdot v.$$

   Show that

   $$\big|B[u,v;t]\big| \le M\|u\|_{H_0^1}\|v\|_{H_0^1}$$

   and

   $$B[u,u;t] \ge \theta\|u\|_{H_0^1}^2 - M\|u\|_{L^2}^2$$

   for some $M, \theta > 0$.

   Hint: for the second one, note that

   $$\|u\|_{L^2}\|u\|_{H_0^1} \le \frac{\varepsilon}{2}\|u\|_{H_0^1}^2 + \frac{1}{2\varepsilon}\|u\|_{L^2}^2$$

   for every $\varepsilon > 0$ and choose ε properly.

2. Establish the energy estimate: for some C,

$$\max_{0\le t\le T}\|u(t,\cdot)\|_{L^2} + \|u\|_{L^2(0,T;H_0^1)} \le C\big[\|f\|_{L^2(0,T;L^2)} + \|g\|_{L^2}\big].$$

Hint: You should start with

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx - \int_\Omega u\Delta u\,dx + \sum_{i=1}^d \int_\Omega b^i\partial_{x_i}u\,dx = \int_\Omega fu\,dx.$$

3. (optional) Also estimate the H^{-1} norm of u_t:

$$\|u_t\|_{L^2(0,T;H^{-1})} \le C\big[\|f\|_{L^2(0,T;L^2)} + \|g\|_{L^2}\big].$$

You can use the following definition: for $\varphi \in \mathcal{C}^2(\bar\Omega)$,

$$\|\varphi\|_{H^{-1}} = \sup_{v\in\mathcal{C}_0^\infty}\frac{\int_\Omega \varphi(x)v(x)\,dx}{\|\varphi\|_{H_0^1}}.$$