HW8

November 20, 2025

Exercise 1 Let $u \in \mathcal{C}^2([0,\infty)\times\mathbb{R})$ solve the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0, & (0,\infty)\times\mathbb{R}, \\ u = g, \;\; u_t = h, & \{t=0\}\times\mathbb{R}. \end{cases}$$

Assume that g, h have compact support. Let

$$k(t) = \frac{1}{2}\int_{-\infty}^{\infty} |u_t(t,x)|^2\,dx, \quad p(t) = \frac{1}{2}\int_{-\infty}^{\infty} |u_x(t,x)|^2\,dx,$$

be the kinetic and potential energy.

1. Show that $k(t) + p(t)$ is constant in t (you do not have to use d'Alembert formula.)
2. Show that $p(t) = k(t)$ when t is large.

Exercise 2 Show that all *spherically symmetric* solutions $u(t,x) = u(t,|x|)$ of the 3d wave equation on the whole space

$$u_{tt}(t,x) = \Delta u(t,x), \quad t>0, \; x \in \mathbb{R}^3,$$

can be written as

$$u(t,x) = \frac{F(|x|-t) + G(|x|+a)}{|x|}.$$

Hint: if $\varphi(x) = \varphi(|x|)$, *then* $\Delta\varphi(|x|) = \varphi''(r) + \frac{2}{r}\varphi'(r)$.

Exercise 3 Let $u \in \mathcal{C}^2(\mathbb{R}^4_+)$ solve

$$\begin{cases} u_{tt} - \Delta u = 0, & t>0, \; x\in\mathbb{R}^3, \\ u(0,x) = \varphi(x), & x\in\mathbb{R}^3, \\ u_t(0,x) = \psi(x), & x\in\mathbb{R}^3, \end{cases}$$

with $\varphi, \psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$. Use the Kirchhoff's formula to show that there exists some constant $C>0$ such that

$$|u(t,x)| \le \frac{C}{t}.$$

Exercise 4 Use separation of variables to solve

$$\begin{cases} u_{tt} - u_{xx} = 0, & x\in(0,\pi), \; t>0, \\ u_x(t,0) = u_x(t,\pi) = 0, & t\ge 0, \\ u(0,x) = 0, \;\; u_t(0,x) = \cos x, & x\in[0,\pi]. \end{cases}$$

Exercise 5 Use separation of variables and Duhamel's principle to solve

$$\begin{cases} u_{tt} - u_{xx} = \sin x, & x \in (0,\pi),\ t > 0, \\ u(t,0) = u(t,\pi) = 0, & t \geq 0, \\ u(0,x) = 0 = u_t(0,x) = 0, & x \in [0,\pi]. \end{cases}$$

Exercise 6 Use separation of variables to solve the following wave equation.

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in (0,\pi),\ t > 0, \\ u(0,x) = u_t(0,\pi) = 0, & x \in [0,\pi], \\ u_x(t,0) = t,\ u_x(t,\pi) = 2t, & t \geq 0. \end{cases}$$

Hint: find a function $v(x)$ with $v'(0) = 1$, $v'(\pi) = 2$ and consider $u(t,x) - tv(x)$.