HW6

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Exercise 1 For $\Omega = B_r(0) \subset \mathbb{R}^2$, let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \quad u = g, & x \in \partial\Omega. \end{cases}$$

where f, g are continuous. Show that u satisfies

$$u(0) = \frac{1}{2\pi r} \int_{\partial\Omega} g(x)\, dS(x) + \frac{1}{2\pi} \int_{\Omega} (\ln r - \ln|x|) f(x)\, dx.$$

Hint: consider

$$\varphi(t) = \frac{1}{2\pi t} \int_{\partial B_t(0)} g(x)\, dS(x) + \frac{1}{2\pi} \int_{B_t(0)} (\ln t - \ln|x|) f(x)\, dx,$$

and show that $\varphi'(t) = 0$, $\lim_{t \to 0+} \varphi(t) = u(0)$.

Exercise 2 Let $v \in \mathcal{C}^2(\Omega)$. We say that Ω is subharmonic if $-\Delta v \leq 0$ in Ω.

1. Show that if v is subharmonic, then for any $B_r(x) \subset \Omega$,

   $$v(x) \leq \fint_{B_r(x)} v(y)\, dy.$$

   Hint: let $\varphi(r) = \fint_{B_r(x)} v(y)\, dy$ *and consider* $\varphi'(r)$.

2. Show that if $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and v is subharmonic, then

   $$\max_{\bar{\Omega}} v(x) = \max_{\partial\Omega} v(x).$$

3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth convex function. Show that if u is harmonic, then $v = \phi(u)$ is subharmonic.

4. Show that $v = |\nabla u|^2$ is subharmonic if u is harmonic.

Exercise 3 Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $u(x) \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u = 1, & \Omega, \\ \quad u = 0, & \partial\Omega. \end{cases}$$

Show that for any $x_0 \in \Omega$,

$$\frac{1}{2d} \min_{x \in \partial\Omega} |x - x_0|^2 \leq u(x_0) \leq \frac{1}{2n} \max_{x \in \partial\Omega} |x - x_0|^2.$$

Hint: consider $v(x) = u(x) - \frac{1}{2d}|x - x_0|^2$.

Exercise 4 Let $\Omega_0 \subset \mathbb{R}^d$ be a bounded domain, and $\Omega \coloneqq \mathbb{R}^d \setminus \overline{\Omega_0}$. Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\partial\Omega)$ satisfy

$$\begin{cases} -\Delta u + c(x)u = 0, & \Omega, \\ \qquad\qquad u = g(x), & \partial\Omega, \\ \lim_{|x|\to\infty} u(x) = \ell \in \mathbb{R}, & \end{cases}$$

where $c(x) \geq 0$ is bounded on any bounded subset of Ω. Show that

$$\sup_{\Omega}|u(x)| \leq \max\big\{|\ell|, \max_{\partial\Omega}|g(x)|\big\}.$$

Hint: obtain an L^∞-estimate on $B_R \setminus \Omega_0$ for any $R > 0$, and then take $R \to \infty$.

Exercise 5 Let $\Omega \subset \mathbb{R}^d$ be bounded. Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u + u^3 - u = 0, & \Omega, \\ u = g, & \partial\Omega. \end{cases}$$

Show that if $\max_{\partial\Omega}|g(x)| \leq 1$, then $\max_{\bar{\Omega}}|u(x)| \leq 1$.

Hint: let $x_0 = \operatorname{argmax}_{x\in\bar{\Omega}} u(x)$; use the fact that

$$u > 1 \implies u^3 - u > 0,$$

to get a contradiction.

Exercise 6 Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u + c(x)u = f(x), & \Omega, \\ \dfrac{\partial u}{\partial n} + \alpha(x)u = 0, & \partial\Omega, \end{cases}$$

where $\alpha(x) \geq 0$ and $c(x) \geq c_0 > 0$. Show that there exists a constant $M = M(c_0)$,

$$\int_\Omega |\nabla u(x)|^2\,dx + \frac{c_0}{2}\int_\Omega |u(x)|^2\,dx + \int_{\partial\Omega} \alpha(x)u^2(x)\,dS(x) \leq M\int_\Omega |f(x)|^2\,dx.$$