HW5

October 14, 2025

Exercise 1 Show that $\Delta u = 0$ is invariant under rotation, that is, if $\Delta u = 0$ and $O \in O(d)$ is a $d \times d$ orthogonal matrix, then

$$v(x) := u(Ox), \quad x \in \mathbb{R}^d$$

also solves $\Delta v = 0$.

Exercise 2 Show that if there exists a solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ to the Neumann problem

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = g, & x \in \partial\Omega, \end{cases}$$

then

$$\int_\Omega f\, dx = -\int_{\partial\Omega} g\, dS.$$

Hint: use integration by parts on $\int_\Omega (\Delta u) v\, dx$ *with* $v \equiv 1$.

Exercise 3 Let $f \in \mathcal{C}^2(\mathbb{R}^3)$ be supported on $B_1(0)$.

1. Use the fundamental solution in $\mathbb{R}^3$ to find a solution to the equation

$$\begin{cases} -\Delta u(x) = f(x), & x \in \mathbb{R}^3, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases}$$

   (You can leave the answer as an integral.)

2. Show that $u(x) \sim \dfrac{c}{|x|}$ for $|x|$ large, and determine the constant c.

3. (Optional) Give a physical interpretation of the result.

Exercise 4 Let $\Omega = \{(x_1, x_2) : x_1, x_2 > 0\}$ be the first quadrant. Use reflection symmetry to find the Green's function in Ω, i.e., for each $y \in \Omega$, solve

$$\begin{cases} -\Delta G(x) = \delta(x - y), & x \in \Omega, \\ G(x) = 0, & x \in \partial\Omega. \end{cases}$$