HW3

September 25, 2025

Recall that the heat kernel is given by

$$G_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0,\ x \in \mathbb{R}^d.$$

Exercise 1

1. Let $\phi \in \mathcal{C}[0,\infty)$ with $\phi(0) = 0$. Show that

$$u(t,x) = \int_0^\infty \big[G_t(x-y) - G_t(x+y)\big]\phi(y)\,dy$$

satisfies $\partial_t u - \partial_{xx} u = 0$ for $t, x > 0$, $u(t,0) \equiv 0$ and

$$\lim_{t\to 0+} u(t,x) = \phi(x), \quad x > 0.$$

Hint: extend $\phi(x)$ to an odd function on $\mathbb{R}$.

2. Find a solution for the half-line heat equation with Neumann boundary condition:

$$\begin{cases} \partial_t u - \Delta u = 0, & t, x > 0, \\ u(0,x) = \phi(x), & x > 0, \\ \partial_x u(t,0) = 0, & t > 0, \end{cases}$$

where $\phi \in \mathcal{C}(\mathbb{R})$ and $\phi'(0) = 0$.

Hint: extend $\phi(x)$ to an even function on $\mathbb{R}$.

Exercise 2 Let $g \in \mathcal{C}^\infty[0,\infty)$ satisfy $g(0) = 0$. Derive that the solution to the heat equation on half-line

$$\begin{cases} \partial_t u = \partial_{xx} u, & t > 0,\ x > 0, \\ u(0,x) = 0, & t = 0,\ x > 0, \\ u(t,0) = g(t), & x = 0,\ t \geq 0, \end{cases}$$

is given by

$$u(t,x) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)\,ds.$$

Hint: let $v(t,x) := u(t,x) - g(t)$; consider the odd extension of v and solve the resulting non-homogeneous heat equation.

Exercise 3 Use Separation of Variables to solve

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0,\pi),\ t > 0, \\ u(0,x) = \sin x, & x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \geq 0. \end{cases}$$

Exercise 4 Use Separation of Variables to solve

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0, \ell),\ t > 0, \\ u(0, x) = x^2(\ell - x)^2, & x \in [0, \ell], \\ \partial_x u(t, 0) = \partial_x u(t, \ell) = 0, & t \geq 0. \end{cases}$$

Exercise 5 Let $u(t, x)$ be the solution to the following initial-Neumann problem obtained via Separation of Variables:

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0, \ell),\ t > 0, \\ u(0, x) = \varphi(x), & x \in [0, \ell], \\ \partial_x u(t, 0) = \partial_x u(t, \ell) = 0, & t \geq 0. \end{cases}$$

Show that $\lim_{t \to \infty} u(t, x)$ exists and find the limit.

Can you give a physical interpretation of this limit?