HW2

September 18, 2025

Exercise 1 Let Ω be a bounded $\mathcal{C}^1$-domain. Show that there is at most one solution $u \in \mathcal{C}^{1,2}((0,\infty)\times\Omega) \cap \mathcal{C}^{0,1}([0,\infty)\times\bar{\Omega})$ that solves

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + f(t,x), & t>0,\ x\in\Omega, \\ \dfrac{\partial u(t,x)}{\partial n} = g(x), & x\in\partial\Omega, \\ u(0,x) = h(x), & x\in\Omega. \end{cases}$$

Hint: for two solutions u_1, u_2, compute $\phi'(t)$ where $\phi(t) = \int_\Omega |\nabla(u_1-u_2)|^2\,dx$.

Exercise 2 Compute the Fourier transform of the following functions defined on $\mathbb{R}$.

1. $$f_1(x) = \begin{cases} 1, & |x|\le A, \\ 0, & |x|>A, \end{cases} \quad A>0.$$

2. $$f_2(x) = \begin{cases} e^{-ax}, & x>0, \\ 0, & x<0, \end{cases} \quad a>0.$$

3. $f_3(x) = e^{-a|x|}$, $a>0$.

4. $f_4(x) = \frac{1}{a^2+x^2}$, $a>0$.

Exercise 3 Let $f\in L^1(\mathbb{R}^d)$. Use Fourier transform to solve the equation

$$-\Delta u(x) + u(x) = f(x), \quad x\in\mathbb{R}^d.$$

Exercise 4

1. Show that
$$\Big[\frac{1}{2}\mathbb{1}_{(-t,t)}(x)\Big]^\wedge = \frac{\sin(2\pi\xi t)}{2\pi\xi}.$$

2. Show that
$$\big[\delta(x-t)+\delta(x+t)\big]^\wedge = 2\cos(2\pi\xi t).$$
You can treat $\delta(x)$ as a function such that $\int \delta(x)f(x)\,dx = f(0)$ for any continuous f.

3. Use Fourier transform to solve the wave equation in $\mathbb{R}^1$:
$$\begin{cases} \partial_{tt}u = \partial_{xx}u, & t>0,\ x\in\mathbb{R}, \\ u(0,x) = \phi(x), & x\in\mathbb{R}, \\ \partial_t(0,x) = \psi(x), & x\in\mathbb{R}. \end{cases}$$