HW11

December 10, 2025

Exercise 1 (Conservation law) Assume that $H(0) = 0$ and $u \in \mathcal{C}_c([0, \infty) \times \mathbb{R})$ is a weak solution to

$$u_t + \big(H(u)\big)_x = 0, \quad u(0, x) = g(x),$$

that is, for every $v \in \mathcal{C}_c^\infty([0, \infty) \times \mathbb{R})$,

$$\int_{\mathbb{R}} dx \int_0^\infty dt \left[uv_t + H(u)v_x\right] + \int_{\mathbb{R}} g(x)v(0, x)\, dx = 0.$$

Show that

$$\int_{-\infty}^{\infty} u(\cdot, t)\, dx = \text{constant},$$

and identify the constant.

Hint: take $v(t, x) = \phi(t)\chi(x)$ where $\chi(x) \approx \mathbb{1}_{[-R,R]}(x)$ for sufficiently large R.

Exercise 2

1. Use the Hopf–Lax formula to solve the Burgers equation

$$u_t + u \cdot u_x = 0, \quad u(0, x) = g(x),$$

with the initial condition

$$g(x) = \begin{cases} 1, & x < -1, \\ 0, & -1 < x < 0, \\ 2, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$$

2. Draw a picture of the characteristics and verify the Rankine–Hugoniot condition along the shocks.