HW10

December 4, 2025

Exercise 1 Let $H : \mathbb{R}^n \to \mathbb{R}$ be convex with

$$L(q) \coloneqq H^*(q) = \sup_{p \in \mathbb{R}^n} \{p \cdot q - H(p)\},$$

where $p \cdot q = p_1 q_1 + \cdots + p_n q_n$ is the dot product in $\mathbb{R}^n$.

1. Show that if $H(p) = \frac{1}{r}|p|^r$, $r \in (1, \infty)$, then $L(q) = \frac{1}{s}|q|^s$, with $r^{-1} + s^{-1} = 1$.
2. Let

   $$H(p) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} p_i p_j + \sum_{i=1}^{n} b_i p_i,$$

   where $A = (a_{ij})$ is symmetric, positive definite. Compute $L(q)$.

Exercise 2 Let $H : \mathbb{R} \to \mathbb{R}$ be convex and $L = H^*$. We say $q \in \partial H(p)$, if

$$H(r) \geq H(p) + q \cdot (r - p), \quad \forall r \in \mathbb{R}.$$

Show that $q \in \partial H(p)$ if and only if $p \in \partial L(q)$, if and only if $p \cdot q = H(p) + L(q)$.

Exercise 3 Let E be a closed subset of $\mathbb{R}^n$. Apply the Hopf–Lax formula to solve the initial value problem

$$\begin{cases} u_t + |\nabla u|^2 = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\ u = 0,\ x \in E, \quad u = +\infty,\ x \notin E, & t = 0. \end{cases}$$

and show that the solution is $u(t, x) = \frac{1}{4t} \operatorname{dist}(x, E)^2$.

Exercise 4 Use the Hopf–Lax formula to show that, if u^i, $i = 1, 2$, solve

$$\begin{cases} u^i_t + H(\partial_x u^i) = 0, & (0, \infty) \times \mathbb{R}, \\ u^i = g^i, & \{t = 0\} \times \mathbb{R}, \end{cases}$$

then the L^∞-contraction holds

$$\sup_{x \in \mathbb{R}} |u^1(t, x) - u^2(t, x)| \leq \sup_{x \in \mathbb{R}} |g^1(x) - g^2(x)|, \quad t > 0.$$