HW1

September 11, 2025

Exercise 1 Solve $\partial_t u + \partial_x u + u = e^{x+2t}$ with initial condition $u(0, x) = 0$.

Exercise 2 Consider the following initial value problem for Burgers equation

$$\begin{cases} \partial_t u + u\partial_x u = 0, \\ u(0,x) = \phi(x) = \begin{cases} 1, & x \le 0, \\ 1-x, & 0 < x \le 1, \\ 0, & x > 1. \end{cases} \end{cases}$$

1. Find the largest time t_s such that all characteristics do not intersect.
2. Find an expression of $u(t, x)$ for $t < t_s$.

Exercise 3 Use the method of characteristics to solve the following PDEs.

1. $x_1 \partial_{x_1} u + x_2 u_{x_2} = 2u$, $u(x_1, 1) = g(x_1)$.
2. $u\partial_{x_1} u + \partial_{x_2} u = 1$, $u(x_1, x_1) = \frac{1}{2}x_1$.

Exercise 4 Suppose that u is smooth and solves $u_t - \Delta u = 0$ in $(0, \infty) \times \mathbb{R}^d$.

1. Show that $u_\lambda(t, x) := u(\lambda^2 t, \lambda x)$ solves the heat equation for every $\lambda \in \mathbb{R}$.
2. Use the above to derive that $v(t, x) := x \cdot \nabla u(t, x) + 2tu_t(t, x)$ also solves the heat equation. *Hint: take derivative in λ.*