Review for midterm

October 28, 2024

1 General concepts

- integration by parts or Divergence Theorem.
- What is a classical solution (for first-order, heat equation, elliptic equation)?
- What are the usual boundary conditions? How do we formulate the boundary conditions?
- Well-posedness: what are existence, uniqueness and stability?

2 First-order PDEs

- Method of characteristics. How to find characteristics? Why are they unique?
- What happens when characteristic intersects? Under what conditions they will not intersect?

3 Heat equation

To solve the equation:

- On the whole space: fundamental solutions
  - How do we use Fourier transform to find the fundamental solutions?
  - How do we use scaling invariance to find the fundamental solutions?
  - How do we use *convolution* with the fundamental solutions to solve the Cauchy problem (initial value problem)? In which sense is the initial condition satisfied? Why is the solution smooth?
  - How to use symmetry to solve heat equation in half-space?
- On bounded domain:
  - principle of superposition. separation of variables. Sturm–Liouville/eigenvalue problem.
  - How to use Fourier series to solve heat equations on bounded intervals?
  - How do different boundary conditions affect the solution?
  - What are Green's functions?
- inhomogeneous problem
  - Duhamel's principle.

  - usage on the whole space (using fundamental solution).
  - usage on a bounded interval (using Fourier series).

Uniqueness and stability

- What is the maximum principle and what is the idea of the proof?
- generalization of maximum principle to other suitable parabolic operators.
- How to use the maximum/comparison principle to prove uniqueness of solution?
- How to formulate L^∞-stability result using the maximum principle.
- Energy estimates and L^2-stability.

4 Elliptic equation

Fundamental solutions and Green's function.

- What are fundamental solutions and Green's functions?
- How to use them to solve Laplace's equation and Poisson's equation?
- What are the fundamental solutions in $\mathbb{R}^d$? How to use them to solve Laplace's equation?
- How to use fundamental solutions to find Green's functions for balls and half-space? What are the Poisson's kernel in both cases?
- How is the boundary condition satisfied for the Laplace equation?
- General properties of Green's function.

Harmonic functions

- vean-value property.
- (strong) maximum principle.
- $\mathcal{C}^\infty$-smoothness of harmonic functions.